

ODE TO L^p NORMS

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ABSTRACT. In this paper we relate the geometry of Banach spaces to the theory of differential equations, apparently in a new way. We will construct Banach function space norms arising as weak solutions to ordinary differential equations (ODE) of first order. This provides as a special case a new way of defining varying exponent L^p spaces, different from the Orlicz type approach. We explain heuristically how the definition of the norm by means of the particular ODE is justified. The resulting class of spaces includes the classical L^p spaces as a special case. It turns out that the duality of these spaces behaves in an anticipated way, similarly as the uniform convexity and uniform smoothness. A noteworthy detail regarding our $L^{p(\cdot)}$ norms is that they satisfy Hölder's inequality properly. We study the arising duality of the ODEs and also the duality of their solutions. We also investigate an ODE-free means of analyzing the norms. Extensions of the definitions to several directions are discussed at the end.

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1. INTRODUCTION

In this paper a novel way will be introduced of defining function space norms by means of weak solutions to ordinary differential equations (ODE). This provides a new perspective for looking at varying exponent L^p spaces. This also leads to looking at the geometry and duality of Banach spaces in terms of the properties of corresponding differential equations.¹

The classical Birnbaum-Orlicz norms were defined in the 1930's and since then there have been various generalizations of these norms to several directions. Notable examples of norms and spaces carry names such as Besov, Luxemburg, Lorentz-Zygmund, Musielak-Orlicz, Nakano, Orlicz and Triebel-Lizorkin, see e.g. [2], [22], [31], [24]. These norms have been recently applied to other areas of mathematics as well as to some real-world applications, see e.g. [1], [5], [12], [7], [28]. Roughly speaking, these norms can be viewed as belonging to a family of derivatives of the Minkowski functional. This kind of approach leads to several varying exponent $L^{p(\cdot)}$ type constructions, e.g. for sequence spaces, Lebesgue spaces, Hardy spaces and Sobolev spaces. There is a vast literature on these topics, see [19], [20], [25] and [27] for samples and further references. There are also other ways of looking at the varying exponent L^p spaces such as the Marcinkiewicz space whose approach differs from the one mentioned above, see [23].

Let us recall that the general Musielak-Orlicz type norms are defined as follows:

$$\|f\| = \inf \left\{ \lambda > 0: \int_{\Omega} \phi \left(\frac{|f(t)|}{\lambda}, t \right) dm(t) \leq 1 \right\}.$$

Here ϕ is a positive function satisfying suitable structural conditions. For instance, $\phi(s, t) = s^{p(t)}$, or $\psi(s, t) = \frac{s^{p(t)}}{p(t)}$, $1 \leq p(\cdot) < \infty$, produces a norm that can be seen as a varying exponent L^p norm. In the latter case we have the *Nakano norm*, which, as it turns out, is of particular interest in this paper.

In contrast, the basic form of the norm that will be introduced here differs considerably from the above-mentioned norms in the sense that it does not arise as a derivation of the Minkowski functional, and it does not apply any norming set of functionals either. In some cases the classes of spaces introduced here do not coincide as sets with any of the classes mentioned above for a given $p: [0, 1] \rightarrow [1, \infty)$ measurable. This is due to obstructions that will become obvious shortly. Also, there appears to be no particular reason to assume, a priori, that the norms are related, although some partial information is available, see Proposition 3.3. However, we are not taking any position here on whether or not the norms introduced here are equivalent in some cases to some of the above-mentioned norms (constant p case excluded).

The above norms enjoy the attractive property of being rearrangement invariant in the sense that applying a measure-preserving transformation $T: \Omega \rightarrow \Omega$ onto such that $\phi(|f(x)|, x) = \phi(|f \circ T(x)|, T(x))$ for a.e. $x \in \Omega$ does not change the value of the norm. However, one may argue that the rearrangement invariance and apparent simplicity of the definition of the norm come with a cost. Namely, the definition of the norm is opaque in the sense that it involves an infimum with an integral formula inequality having rather complicated interdependencies at the binding surface of the feasible set. For instance, by looking at the definition of the

¹The author presented some of the results of this paper in the following function space related meetings in May-June 2014: Edwardsville, Albacete, Mekrijärvi.

norm it is difficult to decide how adding 1_Δ , $\Delta \subsetneq \Omega$, $m(\Delta) > 0$, to f contributes to the norm, even if Δ is in some sense conveniently displaced.

The ‘virtues and vices’ of the norms about to be introduced are mirror images of the ones mentioned above. The ODE driven norms here, in comparison, will typically not be rearrangement invariant in the above sense, and in particular they do not reduce to the above Luxemburg type (e.g. an example after Proposition 3.3, cf. example in [30]). On the other hand, our norms will be ‘localized’ in the sense that one can analyze the (infinitesimal) contribution of a single coordinate to the norm so far, a built-in feature of the construction. To make a point, it is possible to compute these norms by solving the defining ODE numerically for continuous functions f and p . (It is, of course, also straightforward to compute the above infimum numerically, but we stress the fact that the methods needed to solve our first order ODE are linear in nature and elementary.) Thus, our approach to the definition of varying exponent L^p space norms is rather inductive than global.

Next we will discuss the motivating ideas behind the ODE driven norms. The author studied in [30] varying exponent $\ell^{p(\cdot)}$ spaces formed in the following naïve fashion. As usual, we denote by $X \oplus_p Y$ the direct sum of Banach spaces X and Y with the norm given by

$$\|(x, y)\|_{X \oplus_p Y}^p = \|x\|_X^p + \|y\|_Y^p, \quad x \in X, y \in Y, \quad 1 \leq p < \infty.$$

Let $p: \mathbb{N} \rightarrow [1, \infty)$ be a ‘varying exponent’. Define first a 2-dimensional Banach space by $\mathbb{R} \oplus_{p(1)} \mathbb{R}$, then a 3-dimensional one $(\mathbb{R} \oplus_{p(1)} \mathbb{R}) \oplus_{p(2)} \mathbb{R}$ and proceed recursively to obtain n -dimensional spaces

$$(\dots((\mathbb{R} \oplus_{p(1)} \mathbb{R}) \oplus_{p(2)} \mathbb{R}) \oplus_{p(3)} \dots) \oplus_{p(n-1)} \mathbb{R},$$

and, finally, by taking an inverse limit, this yields a space which can be written formally as

$$\dots(\dots((\mathbb{R} \oplus_{p(1)} \mathbb{R}) \oplus_{p(2)} \mathbb{R}) \oplus_{p(3)} \dots) \oplus_{p(n)} \mathbb{R}) \oplus_{p(n+1)} \dots \quad .$$

Thus, this is a space normed by taking a limit of semi-norms corresponding to the n -dimensional spaces above. The recursive construction of the spaces can be regarded trivial at each step, but the end result may exhibit some peculiar properties, depending on the selection of the sequence $(p(n))_{n \in \mathbb{N}}$, see [30]. For instance, it provides an *easy* example of a separable Banach space X with a 1-unconditional basis such that X contains all spaces ℓ^p , $1 \leq p < \infty$, almost isometrically. In any case, this appears a rather natural way of constructing Banach sequence spaces (cf. Kalton et al. [3, 15]).

The main aim of this paper is to study ‘a continuous version’ of the above class of sequence spaces $\ell^{p(\cdot)}$, thus a space of suitable functions $f: [0, 1] \rightarrow \mathbb{R}$, instead of sequences. The idea is somewhat similar here, knowing the norm of f up to a coordinate $0 < t < 1$, i.e. $\|1_{[0,t]}f\|$, and knowing the value $|f(t^+)|$ is sufficient information in predicting the accumulation of the norm right after t , i.e. knowing $\|1_{[0,t+dt]}f\|$. For example, if $f(r) = 0$ for $t < r < s$, then we should have $\|1_{[0,t]}f\| = \|1_{[0,s]}f\|$, and if $|f(t^+)| > 0$, then $\|1_{[0,t]}f\| < \|1_{[0,s]}f\|$, and so on. This intuitive description of the accumulation of the norm is captured by a suitable ODE in such a way that its weak solution, $\varphi_f: [0, 1] \rightarrow [0, \infty)$, shall represent the norm as follows:

$$(1.1) \quad \varphi_f(t) = \|1_{[0,t]}f\|,$$

so that in particular $\varphi_f(0) = 0$ and $\varphi_f(1) = \|f\|$. The above equation (1.1) neatly outlines the overall strategy implemented in the beginning of the paper. The basic idea in accomplishing this and the heuristic motivation appear shortly, see Section 1.2. Differential equations have been previously studied in connection to varying exponent spaces and Sobolev spaces (see. e.g. [6], [9], cf. [26]) but apparently not in the same vein as they arise here.

The required mathematical machinery in this paper is classical, and there is no apparent reason, why this alternative approach could have not been experimented with much earlier. Also, our approach does not lead to excessively technical considerations, so hopefully it is accessible to a wide range of analysts.

1.1. Preliminaries and auxiliary results. We will usually consider the unit interval $[0, 1]$ endowed with the Lebesgue measure m . Here for almost every (a.e.) refers to m -a.e., unless otherwise specified. Denote by L^0 the space of Lebesgue-to-Borel measurable functions on the unit interval. We denote by $\ell^0(\mathbb{N})$ the vector space of sequences of real numbers with point-wise operations. We refer to [4], [8], [20] and [29] for suitable background information.

We will study Carathéodory's weak formulation to ODEs, that is, in the sense of Picard type integral formulation, where solutions are required to be only absolutely continuous. This means that, given an ODE

$$\varphi(0) = x_0, \quad \varphi'(t) = \Theta(\varphi(t), t), \quad \text{for a.e. } t \in [0, 1],$$

we call φ a weak solution in the sense of Carathéodory if φ is absolutely continuous, $t \mapsto \Theta(\varphi(t), t)$ is measurable and

$$\varphi(T) = x_0 + \int_0^T \Theta(\varphi(t), t) \, dt$$

holds for all $T \in [0, 1]$, where the integral is the Lebesgue integral. In what follows, we will refer to Carathéodory's solutions simply as solutions.

Whenever we make a statement about a derivative we implicitly state that it exists. We will write $F \leq G$, involving elements of L^0 , if $F(t) \leq G(t)$ for a.e. $t \in [0, 1]$. We denote the characteristic function or indicator function by 1_A defined by $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ otherwise.

We usually make the following regularity assumption, call it $(*)$, on ODEs Θ : For all $t \in [0, 1)$ and initial value $x_t > 0$ there is $\delta = \delta_{t, x_t} > 0$ such that Θ has a unique solution φ on $[t, t + \delta)$ with initial value $\varphi(t) = x_t$.

Lemma 1.1. *Suppose that Θ as above is non-negative and satisfies $(*)$. Let $t \in [0, 1)$ and $x_t > 0$. Then exactly one of the following holds:*

- (1) *There is a unique solution $\varphi: [t, 1] \rightarrow \mathbb{R}$ with initial value x_t .*
- (2) *There is a unique solution $\varphi: [t, r) \rightarrow \mathbb{R}$, $t < r \leq 1$, with $\lim_{s \rightarrow r} \varphi(s) = \infty$.*

Proof. The non-negativity of Θ gives that the partial solutions are non-decreasing. By the Hausdorff maximal principle we can 'glue together' partial solutions to get a maximal partial solution of the form $\varphi: [t, r) \rightarrow \mathbb{R}$. If $\lim_{s \rightarrow r} \varphi(s) = x_r < \infty$ and $r < 1$, then $(*)$ yields a further extension, contradicting the maximality. \square

Lemma 1.2. *Suppose that there is a non-decreasing function $\psi: [0, 1] \rightarrow [0, \infty)$ satisfying:*

$$\psi(0) = x_0 \geq 0, \quad \psi'(t) \geq \Psi(\psi(t), t) \geq 0, \quad \text{for a.e. } t \in [0, 1].$$

Consider a differential equation

$$\varphi(0) = y_0 \geq 0, \quad \varphi'(t) = \Phi(\varphi(t), t) \geq 0 \quad \text{for a.e. } t \in [0, 1]$$

where Φ satisfies (*), is continuous and decreasing on the first coordinate, $y_0 \leq x_0$ and $\Phi(x, t) \leq \Psi(x, t)$. Then there is a Φ -solution with initial value y_0 .

Proof. According to (*) and the previous lemma there is a maximal partial Φ -solution $\varphi: [0, r) \rightarrow \mathbb{R}$ with $\varphi(0) = y_0$.

Note that φ is dominated by ψ because $\varphi(t) \geq \psi(t)$ implies

$$\varphi'(t) = \Phi(\varphi(t), t) \leq \Phi(\psi(t), t) \leq \Psi(\psi(t), t) \leq \psi'(t).$$

Indeed, here ψ may be non absolutely continuous, and in fact it may even have jump discontinuities, but since it is non-decreasing we obtain the claimed dominance. We conclude that $r = 1$ and $\varphi(1) < \infty$ exists. \square

We also apply a condition, (vii), which can be seen as a strengthening of (*).

Lemma 1.3. Suppose that $\varphi, \psi \in L^0$ are absolutely continuous such that for a.e. $t \in [0, 1]$

$$\varphi(t) \geq \psi(t) \Rightarrow \varphi'(t) \leq \psi'(t).$$

Then $\varphi \leq \psi$.

Proof. Observe that

$$\varphi'(t) \leq (\min(\varphi, \psi))', \quad \text{for a.e. } t \in [0, 1].$$

\square

We will frequently calculate terms of the form $(a^p + b^p)^{\frac{1}{p}}$ where $a, b \geq 0$ and $1 \leq p < \infty$. We will adopt from [30] the following short hand notation for this:

$$a \boxplus_p b = (a^p + b^p)^{\frac{1}{p}}.$$

This defines a commutative semi-group on \mathbb{R}_+ , in particular, the associativity

$$a \boxplus_p (b \boxplus_p c) = (a \boxplus_p b) \boxplus_p c,$$

is useful. In taking a sequence of \boxplus_p or \oplus_p operations we always perform the operations from left to right, unless there are parentheses indicating another order. We will also use the following operation:

$$\bigoplus_{1 \leq i \leq n}^p x_i = x_1 \boxplus_p x_2 \boxplus_p \dots \boxplus_p x_n = \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}}, \quad x_1, \dots, x_n \in \mathbb{R}_+.$$

The space $\ell^{p(\cdot)} \subset \ell^0$, $p: \mathbb{N} \rightarrow [1, \infty)$, consists of those elements (x_n) such that the following limit of a non-decreasing sequence exists and is finite:

$$\lim_{n \rightarrow \infty} (\dots (((|x_1| \boxplus_{p(1)} |x_2|) \boxplus_{p(2)} |x_3|) \boxplus_{p(3)} |x_4|) \boxplus_{p(4)} \dots \boxplus_{p(n-1)} |x_n|) \boxplus_{p(n)} |x_{n+1}|)$$

and the above limit becomes the norm of the space, see [30].

The author is grateful to Pilar Cembranos and José Mendoza for explaining the argument of the following fact in a personal communication.

Proposition 1.4. *Let $1 \leq p \leq r < \infty$ and $y_k = (a_{ij}^{(k)}) \in \ell^r(\ell^p)$, $k \in \mathbb{N}$, with $a_{ij}^{(k)} a_{ij}^{(l)} = 0$ for all $i, j, k, l \in \mathbb{N}$, $k \neq l$. Then*

$$\left\| \sum_{k \in \mathbb{N}} y_k \right\|_{\ell^r(\ell^p)} \leq \left(\bigoplus_{k \in \mathbb{N}}^p \right) \|y_k\|_{\ell^r(\ell^p)}.$$

Proof. Put $a_{ij} = \sum_k a_{ij}^{(k)}$. Then

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} y_k \right\|_{\ell^r(\ell^p)}^p &= \left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{ij}|^p \right)^{\frac{r}{p}} \right)^{\frac{p}{r}} = \left\| \left(\sum_{j=1}^{\infty} |a_{ij}|^p \right)_{i=1}^{\infty} \right\|_{\ell^{\frac{r}{p}}}^{\frac{p}{r}} = \left\| \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{ij}^{(k)}|^p \right)_{i=1}^{\infty} \right\|_{\ell^{\frac{r}{p}}}^{\frac{p}{r}} \\ &\leq \sum_{k=1}^{\infty} \left\| \left(\sum_{j=1}^{\infty} |a_{ij}^{(k)}|^p \right)_{i=1}^{\infty} \right\|_{\ell^{\frac{r}{p}}}^{\frac{p}{r}} = \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{ij}^{(k)}|^p \right)^{\frac{r}{p}} \right)^{\frac{p}{r}} = \sum_{k=1}^{\infty} \|y_k\|_{\ell^r(\ell^p)}^p. \end{aligned}$$

□

This in turn implies the following fact by decomposing the matrix to columns.

Proposition 1.5. *If $1 \leq p \leq r < \infty$ and (x_{ij}) is an infinite matrix of non-negative numbers, then*

$$\bigoplus_{j \in \mathbb{N}}^r \bigoplus_{i \in \mathbb{N}}^p x_{ij} \leq \bigoplus_{i \in \mathbb{N}}^p \bigoplus_{j \in \mathbb{N}}^r x_{ij}.$$

Equivalently, taking the transpose $T: (x_{ij}) \mapsto (x_{ji})$ defines a norm-1 operator $\ell^p(\ell^r) \rightarrow \ell^r(\ell^p)$.

□

The inequality in Proposition 1.5 can be seen as a ‘distributive version’ of the following fact appearing in [30]:

$$a \boxplus_r (b \boxplus_p c) \leq (a \boxplus_r b) \boxplus_p c, \quad 1 \leq p \leq r \leq \infty, \quad a, b, c \in \mathbb{R}_+.$$

Throughout we will use the convention $0^p = 0$ for $p \in \mathbb{R}$.

Suppose that $h: \mathbb{R}^3 \rightarrow \mathbb{R}$ is a function not depending on f and p . Given a function $g(t) := h(f(t), p(t), t)$, we define a differential operator by

$$(\partial^* g)(t) = \frac{d^+}{d\Delta} h(f(t), p(t), t + \Delta) \Big|_{\Delta=0}$$

whenever the right derivative exists. A priori, this operator need not be well defined, but there is no danger of confusion here in practice in using the above shorthand notation.

This operator can be seen as the composition of a usual differential operator and an operator which maps variable functions to their time-stopped versions. Thus we know that the operation is consistent, simply because all the calculations take place in the space of time-stopped functions.

For example, (1.3) below immediately gives

$$(1.2) \quad (\partial^* \varphi(t)^{p(t)})(t_0) = |f(t_0)|^{p(t_0)}.$$

1.2. Arriving at the varying exponent L^p norm ODE. Let us ‘derive’ heuristically our basic differential equation for varying exponent L^p norm. As mentioned in the introduction, we wish to extend the varying exponent $\ell^{p(\cdot)}$ norm in the sense of [30] to continuous setting. Although the motivation for the task here involves the above sequence spaces, we are only required to look at simple structures $X \oplus_p Y$ one at a time due to the infinitesimal nature of the enterprise.

We will assume a Platonist approach on developing the definition of the varying exponent norms here. Thus we wish to find a function space norm following the gist of $\ell^{p(\cdot)}$ space norms. This leads to thought experiments on the right behavior of the function $t \mapsto \|1_{[0,t]}f\|$. In a sense, the resulting ODE will be a very robust one and this allows us to write arguments in this paper in a concise fashion, not paying very much attention on the general theory of the ODEs involved.

Suppose that we have a varying exponent, i.e. a measurable function $p: [0, 1] \rightarrow [1, \infty)$ and $f: [0, 1] \rightarrow \mathbb{R}$ is another measurable function, a possible candidate to lie in the function space. We wish to arrange matters in such a way that we have an absolutely continuous non-decreasing function $\varphi_f: [0, 1] \rightarrow [0, \infty)$ such that

$$\varphi_f(t) = \|1_{[0,t]}f\|, \quad 0 \leq t \leq 1,$$

so $\varphi_f(0) = 0$ and $\varphi_f(1) = \|f\| < \infty$.

For example, in the classical case of L^p spaces with a constant function $f = 1$ and $p = 1, 2, \infty$ we have

$$\varphi_f(t) = t, \quad \varphi_f(t) = \sqrt{t}, \quad \text{and} \quad \varphi_f(t) = 1_{(0,1]}(t),$$

respectively. Here the p -norms are 1 but the profiles differ considerably. The first two solutions are absolutely continuous and the last one is not even continuous.

We will study Carathéodory’s weak formulation to ODEs. It is convenient to work with absolutely continuous solutions, since this way we may apply such usual tools as Fatou’s lemma and Lebesgue’s convergence theorems on the solutions (sometimes implicitly). We are only interested here in Banach lattice norms, therefore φ_f is non-decreasing. In fact, we will require a mildly modified version of Carathéodory’s weak formulation, tailor-made specifically to our setting.

We are aiming at a recursive-like formula for φ_f , in a similar spirit as in [30], so suppose that we have defined the function φ_f up to the interval $[0, t_0]$. Then we are not interested in the values of f and p on $[0, t_0)$, a Markovian property. Suppose, as a thought experiment, that f and p are constant on an interval $[t_0, t_0 + \Delta]$, $\Delta > 0$. Then we should have

$$\begin{aligned} \varphi(t_0 + \Delta) &= (\varphi(t_0)^{p(t_0)} + \Delta |f(t_0)|^{p(t_0)})^{1/p(t_0)}, \\ (1.3) \quad &= \varphi(t_0) \boxplus_{p(t_0)} \Delta^{1/p(t_0)} |f(t_0)| \end{aligned}$$

analogous to the $\ell^{p(\cdot)}$ construction, and actually to the usual L^p norm formula, since

$$\left(\int_0^{t_0+\Delta} |f(s)|^p dm(s) \right)^{\frac{1}{p}} = \left(\int_0^{t_0} |f(s)|^p dm(s) \right)^{\frac{1}{p}} \boxplus_p \left(\int_{t_0}^{t_0+\Delta} |f(t_0)|^p dm(s) \right)^{\frac{1}{p}}$$

where the right-most term is $\Delta^{1/p(t_0)} |f(t_0)|$. Thus, by differentiating (1.3) we find a natural candidate for the norm-determining differential equation:

$$(1.4) \quad \partial^* \varphi(t_0) := \frac{d^+}{d\Delta} \varphi(t_0 + \Delta) \Big|_{\Delta=0} = \frac{|f(t_0)|^{p(t_0)}}{p(t_0)} \varphi(t_0)^{1-p(t_0)}.$$

Here we set $\Delta = 0$, because we are interested in (infinitesimal) increments around t_0 . So, the above equation is right if f and φ are constant on the interval $[t_0, t_0 + \Delta]$ but the equation does *not* concern the values of f , φ and p *beyond* t_0 .

In formulating the differential equation we do not require f or p to be continuous anywhere, but motivated by Lusin's theorem and related considerations we will use the above formula in any case and aim to define φ by

$$(1.5) \quad \varphi(0) = 0, \quad \varphi'(t) = \frac{|f(t)|^{p(t)}}{p(t)} \varphi(t)^{1-p(t)} \quad \text{for a.e. } t \in [0, 1].$$

This formulation has the drawback that $0^{1-p(t)}$ is not defined. Also, it has a trivial solution $\varphi \equiv 0$, regardless of the values of f if we use the convention $0^0 = 0$ and $p \equiv 1$. The behavior of the solutions is difficult to anticipate in the case where $\varphi(t)$ is small and $p(t)$ is large.

To fix these issues, we will use initial values $\varphi(0) = x_0 > 0$ and to correct the error incurred we let $x_0 \searrow 0$. It turns out that the corresponding unique solutions φ_{x_0} decreasingly converge point-wise to φ which again satisfies the same ODE (where applicable). So, this procedure yields a unique solution φ which we will formulate, by slight abuse of notation, as

$$(1.6) \quad \varphi(0) = 0^+, \quad \varphi'(t) = \frac{|f(t)|^{p(t)}}{p(t)} \varphi(t)^{1-p(t)} \quad \text{for a.e. } t \in [0, 1].$$

There is more to the above procedure than merely picking a maximal solution; it turns out that in many situations it is convenient to look at positive-initial-value solutions first.

We define the varying exponent space $L^{p(\cdot)} \subset L^0$ as the space of those functions $f \in L^0$ such that $\varphi_f(1) < \infty$ where φ_f is an absolutely continuous solution to (1.6) and the norm of f will be $\varphi_f(1)$. Shortly we will study the properties of these spaces in an abstract setting more carefully.

The above ODE is a separable one for a constant $p(\cdot) \equiv p$, $1 \leq p < \infty$, and solving it (see (3.4)) yields $\varphi_f(1)^p = \int_0^1 |f(t)|^p dt$, compatible with the classical definition of the L^p norm. If $p(\cdot)$ is locally bounded and $|f(t)|^{p(t)}$ is locally integrable, then Picard iteration performed locally yields a unique solution for each initial value $\varphi(0) = a > 0$. (More precisely, here local means that every point has an open neighborhood such that the function in question has the required property when restricted to the neighborhood and the conclusion holds on some segment $[0, r)$, $0 < r \leq 1$, since, a priori, it is possible that $\varphi(s) \rightarrow \infty$ as $s \rightarrow r$.)

If φ is differentiable in the sense of ∂^* at t_0 , then direct calculation yields

$$(\partial^* \varphi(t)^{p(t)})(t_0) = p(t_0)(\partial^* \varphi)(t_0) \varphi(t_0)^{p(t_0)-1}.$$

Recall (1.2):

$$(\partial^* \varphi(t)^{p(t)})(t_0) = |f(t_0)|^{p(t_0)}.$$

These two lines reproduce (1.4).

2. GENERAL ODE DRIVEN FUNCTIONALS ON FUNCTION CLASSES

Guided by the prototypical norm-determining ODE minted above we will formulate a more general class of Banach lattice function space ODEs. The main point in this section is to isolate the key properties of the norm-determining ODE.

Suppose that $f \in L^0([0, 1])$ and we are dealing with Carathéodory's weak solutions φ_f of the following differential equation:

$$(2.1) \quad \varphi_f(0) = x_0 > 0, \quad \varphi'_f(t) = \Upsilon(\varphi_f(t), |f(t)|, t), \quad \text{for a.e. } t \in [0, 1]$$

where we make the following structural assumptions on Υ :

- (i) $(0, \infty) \times [0, \infty) \rightarrow [0, \infty)$, $(s, x) \mapsto \Upsilon(s, x, t)$ is jointly continuous, non-increasing on s and non-decreasing on x for $t \in [0, 1]$.
- (ii) $(s, x) \mapsto \Upsilon(s, x, t)$ is positively homogeneous and $\Upsilon(s, 0, t) = 0$ for $t \in [0, 1]$.
- (iii) $\lim_{s \rightarrow 0^+} \Upsilon(s, a, t) > 0$, for $a > 0$.
- (iv) If φ_f, φ_g exist then also φ_{f+g} exists in the same sense and $\varphi_{f+g} \leq \varphi_f + \varphi_g$.
- (v) $\varphi_f(1) > \varphi_f(0)$ for small x_0 , unless $f(t) = 0$ a.e.
- (vi) Given $f_n := 1_{A_n}$, $\varphi_{f_n}(1) \rightarrow 0$ implies $m(A_n) \rightarrow 0$ as $n \rightarrow \infty$.
- (vii) If $g \in L^0$ and $\psi: [0, 1] \rightarrow [0, \infty)$ is measurable, increasing and bounded function such that

$$\int_0^1 \Upsilon(\psi(t), |g(t)|, t) dt < \infty$$

then there is a Υ -solution φ_{g, x_0} for any initial value $x_0 > 0$.

- (viii) $|f| \leq |g|$ and the existence of solution to $\Upsilon(\varphi_g(t), |g(t)|, t)$ implies that of to $\Upsilon(\varphi_f(t), |f(t)|, t)$.

Conditions (i)-(iv) imply that $\|f\| := \varphi_f(1)$ is a semi-norm, (v) additionally yields a norm. Under conditions (i)-(iii), (vii) the assumption that Υ is concave on x , call it (iv*), together with some mild assumptions yield (iv). Condition (vi) is used in proving the completeness of the norm. The last condition is in fact implied by (vii) and is included for convenience.

The conditions (iv), (vii) and may fail even for some $L^{p(\cdot)}$ spaces, so that the corresponding functions form a class, rather than a vector space. We will return to this issue.

The differential equation (1.6) discussed in the previous section with

$$\Upsilon(\varphi_f(t), |f(t)|, t) = \frac{|f(t)|^{p(t)}}{p(t)} \varphi_f(t)^{1-p(t)}$$

satisfies the above conditions, except possibly (iv) and (vii) in the unbounded p case. Unfortunately, looking at this ODE it is hard to decide if the triangle inequality holds. However, this can be accomplished by Lemma 3.6 and subsequent observations. Some other examples are

$$\begin{aligned} \Upsilon(\varphi_f(t), |f(t)|, t) &= t|f(t)|^2 \varphi_f(t)^{-1}, \\ \Upsilon(\varphi_f(t), |f(t)|, t) &= \max_{p \in [p_1, p_2]} \frac{|f(t)|^p}{p} \varphi_f(t)^{1-p}, \end{aligned}$$

$1 \leq p_1 < p_2 < \infty$, and

$$\Upsilon(\varphi_f(t), |f(t)|, t) = \max(|f(t)| - \varphi_f(t), 0).$$

In the last case it is easy to check that

$$(2.2) \quad \varphi_{f+g}(t) \geq \varphi_f(t) + \varphi_g(t) \implies \varphi'_{f+g}(t) \leq \varphi'_f(t) + \varphi'_g(t),$$

so that (iv) holds.

Some of the above assumptions are self-explanatory. The assumption that Υ is decreasing on s requires the most attention. From the geometry of the norm point of view, it contributes to the validity of the triangle inequality, as in (2.2), and hence

to the convexity of the unit ball. It also tends to ‘flatten’ the unit ball, similarly as in the case of L^p spaces with $p < \infty$ large. Therefore it can be viewed to have a smoothening effect on the norm as well, as opposed to the extremal non-smooth case $p(\cdot) \equiv 1$ where $s \mapsto \Upsilon(s, x, t)$ is a constant, since $\varphi(t)^{1-p(t)} \equiv 1$.

From the point of view of differential equations, the assumption that Υ is non-increasing on s also has a remarkably stabilizing effect on the solutions φ . The corresponding differential equation exhibits a mean reverting phenomenon as follows. Suppose that φ and ψ are different solutions, possibly with different initial values, say $x_0, y_0 > 0$, respectively. Then $\varphi'(t) \leq \psi'(t)$ if $\varphi(t) \geq \psi(t)$ and vice versa. Recall Lemma 1.3. Then we easily obtain the following facts:

(2.3)

- (1) If $\varphi(t_0) = \psi(t_0)$ for some t_0 then it follows that $\varphi(t) = \psi(t)$ for $t > t_0$.
- (2) If $x_0 \geq y_0$, then $\varphi \geq \psi$.
- (3) $|\varphi(t) - \psi(t)| \leq |x_0 - y_0|$ for all $t \in [0, 1]$.
- (4) If a solution exists for one initial value $x_0 > 0$, then it exists for all initial values $y_0 > x_0$.
- (5) The solution corresponding to the initial condition $\varphi(0) = 0^+$ exists and is unique if the solutions exist separately for all initial values $x_0 > 0$.

The existence of the particular initial condition solution is seen by using the facts that Υ is continuous with respect to (s, x) , non-increasing with respect to s , and then using the Lebesgue’s monotone convergence theorem on the weak formulation as follows:

$$\varphi_{x_0}(T) = x_0 + \int_0^T \Upsilon(\varphi_{x_0}(t), |f(t)|, t) dt \searrow \int_0^T \Upsilon(\varphi(t), |f(t)|, t) dt = \varphi(T)$$

where $\varphi_{x_0}(t) \searrow \varphi(t)$ and $\Upsilon(\varphi_{x_0}(t), |f(t)|, t) \nearrow \Upsilon(\varphi(t), |f(t)|, t)$ as $x_0 \searrow 0$.

This means that we may refer unambiguously to 0^+ -initial value solutions.

2.1. ODE determined Banach function spaces. Suppose that Υ is a structure function satisfying the above conditions (i) – (vii). We denote by L^Υ the set of all functions $f \in L^0$ which admit a solution φ_f :

$$\varphi_f(0) = 0^+, \quad \varphi'_f(t) = \Upsilon(\varphi_f(t), |f(t)|, t), \quad \text{for a.e. } t \in [0, 1]$$

such that $\varphi_f(1) < \infty$.

As customary, we will identify functions $f \in L^\Upsilon$ which coincide a.e. We consider $L^\Upsilon \subset L^0$ endowed with point-wise linear operations and the partial order $f \leq g$ (of a.e. dominance).

In case L^Υ is a linear space, we will denote the functional $L^\Upsilon \rightarrow \mathbb{R}$, $f \mapsto \varphi_f(1)$ as a norm $\|f\| = \|f\|_{L^\Upsilon} = \varphi_f(1)$.

Theorem 2.1. *Assume that L^Υ is as above. Then $\|f\| := \varphi_f(1)$ defines a norm and $(L^\Upsilon, \|\cdot\|, \leq)$ is a Banach lattice.*

Proof of Theorem 2.1. Note that if $f \in L^\Upsilon$ is not zero a.e., then $\lim_{x_0 \rightarrow 0^+} \varphi'_{f, x_0}(t) > 0$ in subset of positive measure, thus $\varphi_f(1) - \varphi_f(0) > 0$, since Υ is decreasing on φ . It is clear by using assumption (ii) that if $\varphi_f(1) < \infty$, $\lambda \geq 0$ above, then $\varphi_{\lambda f}(t) = \lambda \varphi_f(t)$, so that $\lambda f \in L^\Upsilon$. Take $f, g \in L^\Upsilon$. Then $f + g \in L^\Upsilon$ by (iv) and

$$\|f + g\| \leq \|f\| + \|g\|.$$

Digression: Note that in the case of (iv*) we obtain by (i) and (ii) that

$$\begin{aligned} \Upsilon(\varphi_f(t) + \varphi_g(t), |f(t) + g(t)|, t) &\leq \Upsilon(\varphi_f(t) + \varphi_g(t), |f(t)|, t) + \Upsilon(\varphi_f(t) + \varphi_g(t), |g(t)|, t) \\ &\leq \Upsilon(\varphi_f(t), |f(t)|, t) + \Upsilon(\varphi_g(t), |g(t)|, t) \quad \text{for a.e. } t \in [0, 1]. \end{aligned}$$

Thus, given solutions φ_f and φ_g with $\varphi_f(0) = \varphi_g(0) = x_0 > 0$ we have a majorizing function $\varphi_f + \varphi_g$ for the differential equation

$$\varphi_{f+g}(0) = x_0, \quad \varphi'_{f+g}(t) = \Upsilon(\varphi_{f+g}(t), |(f+g)(t)|, t) \quad \text{for a.e. } t \in [0, 1].$$

By Lemma 1.2 and some mild additional assumptions on the space we obtain a maximal solution φ_{f+g} which is necessarily majorized by $\varphi_f + \varphi_g$. By letting $x_0 \rightarrow 0^+$ we observe the triangle inequality and that the solutions for $f + g$ satisfy (iii).

To continue the main argument, it is clear by the construction of the norm that $\| |f| \| = \|f\|$. Moreover, $|f| \leq |g|$ implies $\|f\| \leq \|g\|$, provided that both the functions are in the space. The above monotonicity of the norm is seen as follows: suppose that φ_f and φ_g are the solutions and $\varphi_f(t) \geq \varphi_g(t)$ on $t \in A \subset [0, 1]$, then

$$(2.4) \quad \Upsilon(\varphi_f(t), |f(t)|, t) \leq \Upsilon(\varphi_g(t), |g(t)|, t)$$

for a.e. $t \in A$, since Υ is decreasing on the first coordinate and increasing on the second. Thus Lemma 1.3 applies and L^Υ has a lattice norm.

In fact, for every measurable $A \subset [0, 1]$ the mapping $R: f \mapsto f - 1_A 2f$ defines an isometric reflection operator which induces a bicontractive projection.

Next we will verify the completeness of the norm. Let $(f_n) \subset L^\Upsilon$ be a Cauchy sequence. Let $f(t) = \lim_{n \rightarrow \infty} f_n(t)$ if the limit exists and $f(t) = 0$ otherwise. In order to prove that $\|f_n - f\| \rightarrow 0$ as $n \rightarrow \infty$ it suffices to prove that $f \in L^\Upsilon$ and for any subsequence n_k there is a further subsequence n_{k_j} such that $\|f_{n_{k_j}} - f\| \rightarrow 0$ as $j \rightarrow \infty$.

Towards this, fix a subsequence n_k . Let

$$A_{n,m,\varepsilon} = \{t \in [0, 1] : |f_n(t) - f_m(t)| > \varepsilon\} \quad n, m \in \mathbb{N}, \varepsilon > 0.$$

According to (v) and (2.4) we obtain that $a_{i,\varepsilon} := \sup_{j \geq i} m(A_{n_i, n_j, \varepsilon}) \rightarrow 0$ as $i, j \rightarrow \infty$, since

$$\varepsilon \|1_{A_{n,m,\varepsilon}}\| \leq \|f_n(t) - f_m(t)\| \rightarrow 0$$

as $n, m \rightarrow \infty$. Thus we may construct a subsequence n_{k_j} and $\varepsilon_j \rightarrow 0$ such that $\sum_j a_{n_{k_j}, \varepsilon_j} < \infty$ and $\sum_j \|f_{n_{k_{j+1}}} - f_{n_{k_j}}\| < \infty$.

(If n_k and m_k are subsequences involving a Cauchy sequence, then the above double subsequence can be additionally chosen in such a way that it contains infinitely many members of both subsequences.)

Put $A_{n_{k_j}} = \bigcup_{i \geq j} A_{n_{k_j}, n_{k_i}, \varepsilon_i}$ and note that this is a decreasing sequence of sets whose measures tend to 0. Thus $f(t) = \lim_{j \rightarrow \infty} f_{n_{k_j}}(t)$ exists a.e.

Note that $\varphi_j := \varphi_{f_{n_{k_j}}}$ is Cauchy on $C([0, 1])$. Indeed, assumption $\varphi_{j+1}(t) - \varphi_j(t) = \delta$ and the triangle inequality for any given t (involving functions of the type $1_{[0,t]}g$) yield that $\varphi_{(f_{n_{k_{j+1}}} - f_{n_{k_j}})}(t) \geq \delta$, so that $\varphi_{(f_{n_{k_{j+1}}} - f_{n_{k_j}})}(1) \geq \delta$. Thus $\varphi = \lim_{j \rightarrow \infty} \varphi_j$ exists.

Therefore the continuity of Υ gives that $\Upsilon(\varphi_j(t), |f_j(t)|, t) \rightarrow \Upsilon(\varphi(t), |f(t)|, t)$ for a.e. t as $j \rightarrow \infty$. Fatou's lemma gives that $\int_0^1 \Upsilon(\varphi(t), |f(t)|, t) dt < \infty$ exists. Thus, by (vii) there is a Υ -solution φ_f . Hence $f \in L^\Upsilon$.

Finally, we wish to verify that

$$\|f - f_{n_{k_j}}\| = \left\| \text{a.e.} - \lim_{m \rightarrow \infty} \sum_{i=j}^m (f_{n_{k_{i+1}}} - f_{n_{k_i}}) \right\| \leq \sum_{i \geq j} \|f_{n_{k_{i+1}}} - f_{n_{k_i}}\| \rightarrow 0$$

as $j \rightarrow \infty$.

Let $g_0 = |f_{n_{k_1}}|$ and $g_j = |f_{n_{k_{j+1}}} - f_{n_{k_j}}|$ for $j \in \mathbb{N}$. Put $h_i = \sum_{j=0}^i g_j$, $i \in \{1, \dots, \infty\}$. By using (iv), the construction of the double subsequence and the same (t) triangle inequality argument as above we obtain that $\varphi_{h_i} \nearrow \varphi_{h_\infty}$ uniformly as $i \rightarrow \infty$.

By Lebesgue's monotone convergence theorem we obtain that

$$\Upsilon(\varphi_{h_\infty}(t), h_i(t), t) \rightarrow \Upsilon(\varphi_{h_\infty}(t), h_\infty(t), t)$$

in L^1 as $i \rightarrow \infty$.

By (vii) and (viii) we see that for any $x_0 > 0$ there is a solution $\varphi_{(f-f_{n_{k_j}}), x_0}$ with

$$\varphi'_{(f-f_{n_{k_j}}), x_0} = \Upsilon(\varphi_{(f-f_{n_{k_j}}), x_0}, |f - f_{n_{k_j}}|, t) \leq \Upsilon(\varphi_{h_\infty}(t), h_\infty(t), t),$$

since $|f - f_{n_{k_j}}| \leq h_\infty$. We may select as small as we wish an initial value. Same applies to $\varphi(1)$ by repeating the same arguments starting with the index k_j , with a large value of j , in place of k_1 . This means that for each $\varepsilon > 0$ there is j_0 such that the corresponding 0^+ -initial value solution $\varphi_{(f-f_{n_{k_j}}), j \geq j_0}$, is bounded from above by ε . Thus $\|f - f_{n_{k_j}}\| \rightarrow 0$ as $j \rightarrow \infty$.

Since (f_n) was Cauchy, we see that f does not depend on the selection of (n_k) . \square

3. BASIC PROPERTIES OF $L^{p(\cdot)}$ SPACES

In this section we will study only spaces of the type $L^{p(\cdot)}$ with $p: [0, 1] \rightarrow [1, \infty)$ measurable. Some of the unbounded functions $p(\cdot)$ actually produce a class of functions, rather than a linear space (see Example 3.4). We will first restrict our considerations to those $L^{p(\cdot)}$ classes which are Banach space (see Theorem 3.13 below). The norms of these spaces were described in the introductory part.

Proposition 3.1 (Hölder). *Suppose that $f \in L^{p_1(\cdot)}$ and $g \in L^{p_2(\cdot)}$, $p_1^*(\cdot) = p_2(\cdot)$. Then they satisfy Hölder's inequality:*

$$\int_0^1 |f(t)g(t)| dt \leq \|f\|_{p_1(\cdot)} \|g\|_{p_2(\cdot)}.$$

Proof. We obtain from the usual Hölder's inequality (for a suitable atomic measure μ with 2 atoms) that

$$\begin{aligned} (3.1) \quad & \int_0^{t+\Delta} |f(s)g(s)| ds \\ & \leq (\varphi_f(t)^{p(t)} + \Delta |f(t)|^{p(t)})^{1/p(t)} (\varphi_g^*(t)^{p^*(t)} + \Delta |g(t)|^{p^*(t)})^{1/p^*(t)} \end{aligned}$$

provided that

$$(3.2) \quad \int_0^t |f(s)g(s)| ds \leq \varphi_f(t) \varphi_g^*(t)$$

holds and f , g and p are constant on $[t, t + \Delta]$. Differentiating with respect to Δ on both sides and setting $\Delta = 0$ then yields $|f(t)g(t)| \leq \partial^*(\varphi_f \varphi_g^*)(t)$ for a.e. t . Note that

$$(\varphi_f \varphi_g^*)' = \varphi_f' \varphi_g^* + \varphi_f \varphi_g^{*'} = (\partial^* \varphi_f) \varphi_g^* + \varphi_f (\partial^* \varphi_g^*) = \partial^*(\varphi_f \varphi_g^*).$$

The inequality (3.2) can be arranged by first choosing positive initial values. \square

Alternatively, one may resort to Remark 3.10.

3.1. Building blocks and estimates. We may define simple semi-norms as follows. First we define a very simple semi-norm by the formula

$$|f|_{p,\mu}^p = \int |f|^p d\mu$$

where μ is a Lebesgue measure with support restricted to a measurable subset of $[0, 1]$. If $p_i \in [1, \infty)$ and $\sup \text{supp}(\mu_i) \leq \inf \text{supp}(\mu_{i+1})$, $1 \leq i \leq n-1$, then we may define a composite semi-norm as follows

$$\begin{aligned} \|f\|_{(\dots(L^{p_1}(\mu_1) \oplus_{r_2} L^{p_2}(\mu_2)) \oplus_{r_3} \dots \oplus_{r_{n-1}} L^{p_{n-1}}(\mu_{n-1})) \oplus_{r_n} L^{p_n}(\mu_n)} \\ := (\dots(|f|_{p_1, \mu_1} \boxplus_{r_2} |f|_{p_2, \mu_2}) \boxplus_{r_3} \dots \boxplus_{r_{n-1}} |f|_{p_{n-1}, \mu_{n-1}}) \boxplus_{r_n} |f|_{p_n, \mu_n}. \end{aligned}$$

For example, if $p(\cdot) \equiv p_1$ on $[0, t_0)$ and $p(\cdot) \equiv p_2 = r_2$ on $[t_0, 1]$ and $f \in L^{p(\cdot)}$ then $\|f\|_{L^{p(\cdot)}} = \|f\|_{L^{p_1}(\mu_1) \oplus_{p_2} L^{p_2}(\mu_2)}$ where $\text{supp}(\mu_1) = [0, t_0]$ and $\text{supp}(\mu_2) = [t_0, 1]$ (see subsequent Lemma 3.5).

It is easy to see that then $\|\mathbf{1}\|_{p(\cdot)} \rightarrow 2$ as $p_1 \rightarrow \infty$, $t_0 \rightarrow 0^+$ and $p_2 = 1$. This is perhaps surprising, since always $\|\mathbf{1}\|_p = 1$ in the constant p case. We may also reverse the above example as follows, letting above $f_{t_0} \equiv 1/t_0$ on $[0, t_0)$ and $f_{t_0} \equiv 1$ on $[t_0, 1]$ with $p_1 = 1$ and $r_2 = p_2 \rightarrow \infty$ and $t_0 \rightarrow 0^+$ yields that $\|f_{t_0}\|_{p(\cdot)} \rightarrow 1$ whereas $\|f_{t_0}\|_1 \rightarrow 2$.

We suspect that the above examples are typical in the sense that

$$\frac{1}{2} \|f\|_1 \leq \|f\|_{p(\cdot)} \leq 2 \|f\|_\infty$$

should always hold (so that constant 2 would be the best possible according to the above examples). In any case, the above inequalities hold with other constants in place of 2. Namely, suppose that $\varphi_f(t_0) = \|f\|_\infty$. Then

$$\varphi_f(t_0) = \|f\|_\infty, \quad \varphi_f'(t) = \frac{|f(t)|^{p(t)}}{p(t)} \varphi_f(t)^{1-p(t)} \quad \text{for a.e. } t_0 \leq t \leq 1$$

yields

$$\varphi_f'(t) \leq \varphi_f(t), \quad \text{for a.e. } t_0 \leq t \leq 1.$$

Observe that $\varphi_f(1) \leq y(1)$ where y is the solution to $y' = y$ with $y(0) = \|f\|_\infty$, that is, $y(x) = \|f\|_\infty e^x$.

Let $a \in (1, 2)$ be the solution to $a^a = e$. This satisfies that $\frac{b^x}{x}$ is increasing on $x \geq 1$ for all $b > a$.

Proposition 3.2. *The following inequalities hold whenever defined:*

- (1) $\frac{1}{1+a} \|1_{p(\cdot) \geq p} f\|_p \leq \|1_{p(\cdot) \geq p} f\|_{p(\cdot)},$
- (2) $\frac{1}{1+ae} \|1_{p_1(\cdot) \leq p_2(\cdot)} f\|_{p_1(\cdot)} \leq \|1_{p_1(\cdot) \leq p_2(\cdot)} f\|_{p_2(\cdot)},$
- (3) $\|f\|_{p(\cdot)} \leq e \|f\|_\infty.$

Proof. The last inequality was already proved and we will verify the middle inequality which is the most complicated one.

Suppose that $p_1(\cdot) \leq p_2(\cdot)$ and $f \in L^{p_1(\cdot)}$, with $\|f\|_{p_1(\cdot)} = 1 + ae$.

We are mainly interested in excluding the case where $\varphi_{p_2(\cdot),f}(1) < 1$, so suppose that $\varphi_{p_2(\cdot),f}(1) \leq 1$. Let $[r, 1]$ be a maximal interval such that $\varphi_{p_2(\cdot),f}(t) \leq \varphi_{p_1(\cdot),f}(t)$ (and $\varphi_{p_2(\cdot),f}(t) \leq 1$) on it. Let $A \subset [0, 1]$ be the set where $|f(t)| > a$. On $[r, 1] \cap A$ we have

$$\frac{|f(t)|^{p_2(t)}}{p_2(t)} \varphi_{p_2(\cdot),f}(t)^{1-p_2(t)} \geq \frac{|f(t)|^{p_1(t)}}{p_1(t)} \varphi_{p_1(\cdot),f}(t)^{1-p_1(t)}.$$

Indeed, in this set we have

$$\frac{|f(t)|^{p_2(t)}}{p_2(t)} \geq \frac{|f(t)|^{p_1(t)}}{p_1(t)}$$

and

$$\varphi_{p_2(\cdot),f}(t)^{1-p_2(t)} \geq \varphi_{p_2(\cdot),f}(t)^{1-p_1(t)} \geq \varphi_{p_1(\cdot),f}(t)^{1-p_1(t)}.$$

Thus

$$\begin{aligned} & \varphi_{p_1(\cdot),f}(1) - \varphi_{p_2(\cdot),f}(1) \\ &= \int_r^1 \frac{|f(t)|^{p_1(t)}}{p_1(t)} \varphi_{p_1(\cdot),f}(t)^{1-p_1(t)} - \frac{|f(t)|^{p_2(t)}}{p_2(t)} \varphi_{p_2(\cdot),f}(t)^{1-p_2(t)} dt \\ &= \int_{[r,1] \cap A} \frac{|f(t)|^{p_1(t)}}{p_1(t)} \varphi_{p_1(\cdot),f}(t)^{1-p_1(t)} - \frac{|f(t)|^{p_2(t)}}{p_2(t)} \varphi_{p_2(\cdot),f}(t)^{1-p_2(t)} dt \\ &\quad + \int_{[r,1] \setminus A} \frac{|f(t)|^{p_1(t)}}{p_1(t)} \varphi_{p_1(\cdot),f}(t)^{1-p_1(t)} - \frac{|f(t)|^{p_2(t)}}{p_2(t)} \varphi_{p_2(\cdot),f}(t)^{1-p_2(t)} dt \\ &\leq \int_{[r,1] \setminus A} \frac{|f(t)|^{p_1(t)}}{p_1(t)} \varphi_{p_1(\cdot),f}(t)^{1-p_1(t)} - \frac{|f(t)|^{p_2(t)}}{p_2(t)} \varphi_{p_2(\cdot),f}(t)^{1-p_2(t)} dt \\ &\leq \int_{[r,1] \setminus A} \frac{|f(t)|^{p_1(t)}}{p_1(t)} \varphi_{p_1(\cdot),f}(t)^{1-p_1(t)} dt \leq \|1_{[r,1] \setminus A} f\|_{p_1(\cdot)} \leq \|a 1_{[r,1] \setminus A}\|_{p_1(\cdot)} \\ &\leq \|a 1_{[0,1]}\|_{p_1(\cdot)} \leq e \|a 1_{[0,1]}\|_{\infty} = ae. \end{aligned}$$

Thus $\varphi_{p_2(\cdot),f}(1) \geq (1 + ae) - ae = 1$. \square

The following fact connects the investigated varying exponent norm to the Nakano $L^{p(\cdot)}$ norms

$$|||g|||_{p(\cdot)} := \inf \left\{ \lambda > 0 : \int \frac{1}{p(t)} \left(\frac{|g(t)|}{\lambda} \right)^{p(t)} dt \leq 1 \right\}.$$

Proposition 3.3. *Let $p \in L^0$, $f \in L^{p(\cdot)}$ (ODE-driven). Then*

$$|||f|||_{p(\cdot)} \leq \|f\|_{p(\cdot)} \leq 2 |||f|||_{p(\cdot)}.$$

Proof. To prove the left hand estimate it suffices to check that if $\lambda = \|f\|_{p(\cdot)}$, then

$$\int_0^1 \frac{1}{p(t)} \left(\frac{|g(t)|}{\lambda} \right)^{p(t)} dt \leq 1. \text{ So, suppose that } 0 < \varphi_f(1) = \lambda, \text{ then}$$

$$\varphi'_f(t) \geq \frac{|f(t)|^{p(t)}}{p(t)} \lambda^{1-p(t)},$$

(with strict inequality in a set of positive measure if $f \neq 0$), so that

$$\lambda = \varphi_f(1) \geq \int_0^1 \frac{|f(t)|^{p(t)}}{p(t)} \lambda^{1-p(t)} dt = \int_0^1 \lambda \frac{1}{p(t)} \left(\frac{|f(t)|}{\lambda} \right)^{p(t)} dt.$$

This is equivalent to

$$\int \frac{1}{p(t)} \left(\frac{|f(t)|}{\lambda} \right)^{p(t)} dt \leq 1.$$

To check the latter inequality, we may restrict to the case $\|f\|_{p(\cdot)} = 1$ by the positive homogeneity of the norms. If $\|f\|_{p(\cdot)} \leq 1$ then we have the claim, so assume that $0 < t_0 < 1$ is such that $\varphi_f(t_0) = 1$. Then

$$\varphi'_f(t) \leq \frac{|f(t)|^{p(t)}}{p(t)}, \quad \text{for a.e. } t \in [t_0, 1].$$

Thus

$$\varphi_f(1) \leq 1 + \int_{t_0}^1 \frac{|f(t)|^{p(t)}}{p(t)} dt \leq 1 + \int_0^1 \frac{|f(t)|^{p(t)}}{p(t)} dt = 1 + \|f\|_{p(\cdot)} = 2.$$

□

Thus the above Nakano norms are equivalent to the ODE-driven norms considered here. However, these norms do not coincide in general. For example, if $p_1(\cdot)$ is 1 on $[0, \frac{1}{2})$ and 2 on $[\frac{1}{2}, 1]$ and $p_2(\cdot)$ is defined in the opposite way, then $\|f\|_{p_1(\cdot)} = \|f\|_{p_2(\cdot)}$, in the case of Nakano norms (or Musielak-Orlicz norms), for any f with $f(s) = f(\frac{1}{2} + s)$ for $0 \leq s < \frac{1}{2}$. The same rearrangement invariance does not hold for the investigated $\|\cdot\|_{p(\cdot)}$ -norms. Indeed, for the constant function $\mathbf{1}$ we have

$$\|\mathbf{1}\|_{p_1(\cdot)} = \sqrt{\left(\frac{1}{2}\right)^2 + \frac{1}{2}} = \frac{\sqrt{3}}{2} \approx 0.866 < 1.207 \approx \frac{1}{\sqrt{2}} + \frac{1}{2} = \|\mathbf{1}\|_{p_2(\cdot)}.$$

We can use the above ideas to construct counterexamples as well.

Example 3.4. Let $p: [0, 1] \rightarrow [1, \infty)$ be a function defined by

$$\frac{1}{p(t)} \left(\frac{2}{3} \right)^{1-p(t)} = \frac{1}{t - \frac{1}{2}}$$

if $t \in (\frac{1}{2}, 1]$ and 1 otherwise. Then the constant function $f = \mathbf{1}$ is *not* in $L^{p(\cdot)}$.

Assuming the contrary, clearly $\varphi_f(\frac{1}{2})$ would be $\frac{1}{2}$. Suppose that $t_0 > \frac{1}{2}$ is a point such that $\varphi_f(t) \leq \frac{2}{3}$ for $\frac{1}{2} \leq t \leq t_0$. Then we should have

$$\frac{2}{3} - \frac{1}{2} \geq \varphi_f(t_0) - \varphi_f\left(\frac{1}{2}\right) = \int_{\frac{1}{2}}^{t_0} \varphi'_f dt \geq \int_{\frac{1}{2}}^{t_0} \frac{1}{t - \frac{1}{2}} dt = \infty,$$

contradicting the assumption that f was in the class, that is, having an absolutely continuous solution φ_f . However, if we are allowed initial values $x_0 \geq \frac{1}{2}$, then we have nice corresponding solutions.

Also note that $\mathbf{1}_{[0, \frac{1}{2}]} + \mathbf{1}_{[0, 1]} \in L^{p(\cdot)}$. This means that in general the $L^{p(\cdot)}$ class need not be an ideal as a function class (cf. Banach lattice theory), i.e. $g \in L^{p(\cdot)}$, $f \in L^0$, $f \leq g$, does not imply $f \in L^{p(\cdot)}$.

In the above example, we have $\left(1_{[0, \frac{1}{2}]} + 1_{[0, 1]}\right), 1_{[0, \frac{1}{2}]} \in L^{p(\cdot)}$ and $\left(1_{[0, \frac{1}{2}]} + 1_{[0, 1]}\right) - 1_{[0, \frac{1}{2}]} = 1_{[0, 1]} \notin L^{p(\cdot)}$. This shows that for some $p(\cdot)$ the class $L^{p(\cdot)}$ fails to be a linear space. This example is a manifestation of the principle that the higher the value of φ , the more stable the differential equation becomes, *ceteris paribus*. This is simply the consequence of the fact that Υ is decreasing on the first variable, a matter discussed in Section 2.

3.2. Transcending from discrete to continuous state. Consider simple semi-norms of the type

$$\|f\|_N = \|f\|_{(\dots(L^{p_1}(\mu_1) \oplus_{r_2} L^{p_2}(\mu_2)) \oplus_{r_3} \dots) \oplus_{r_n} L^{p_n}(\mu_n)}$$

where $\sup \text{supp}(\mu_i) \leq \inf \text{supp}(\mu_{i+1})$, $r_{i+1} \geq p_{i+1}$, and denote this collection by \mathcal{N} . We say that a semi-norm is of *standard form* if $r_{i+1} = p_{i+1}$ for all $i \in \{1, \dots, n-1\}$. In this case we may, in a sense, extend the elements N of \mathcal{N} to a $L^{\tilde{p}(\cdot)}$ norm by putting $\tilde{p}(t) = p_i$ for $t \in \text{supp}(\mu_i)$ and $\tilde{p}(t) = 1$ otherwise (and this extension is unique). If $\bigcup_i \text{supp}(\mu_i) = [0, 1]$ then a corresponding standard form norm $N \in \mathcal{N}$ satisfies $\|f\|_N = \|f\|_{\tilde{p}(\cdot)}$ by subsequent Lemma 3.5.

Observe that the semi-norms are decreasing on r :s and increasing on p :s. Consider point-wise intervals $[p_t, r_t]$ as follows: $p_t = p_{i+1}$ and $r_t = r_{i+1}$ on the support of μ_{i+1} . We denote by $N \leq p(\cdot)$ whenever $p(t) \in [p_t, r_t]$ for all t such that the interval is defined.

We may define a partial order on \mathcal{N} by setting $N \leq M$ if the partition each of the supports of the measures corresponding to N is a union of supports of the measures corresponding to M and $[p_{M,t}, r_{M,t}] \subset [p_{N,t}, r_{N,t}]$ for each t such that the left hand interval is defined. We note that

$$\mathcal{N}_{\leq p(\cdot)} := \{N \in \mathcal{N} : N \leq p(\cdot)\}$$

is a directed poset, there are natural upper bound and lower bound operations yielding for each $N, M \in \mathcal{N}$ the bounds $N \vee M$ and $N \wedge M$. These are obtained by taking a coarsest common refinement (resp. finest common coarsening) of the supports of measures and then taking $p_t = \max(p_{N,t}, p_{M,t})$ and $r_t = \min(r_{N,t}, r_{M,t})$ (resp. $p_t = \min(p_{N,t}, p_{M,t})$ and $r_t = \max(r_{N,t}, r_{M,t})$) applied on the respective domains.

We may define a functional as follows:

$$(3.3) \quad \rho(f) := \limsup_{N \nearrow \mathcal{N}_{\leq p(\cdot)}} \|f\|_N.$$

It turns out that in some cases the functions $f \in L^0$ with $\rho(f) < \infty$ form a normed space.

We will connect the above limiting process of semi-norms to ODEs. In doing this we are required to use initial values for the ODEs, and, consequently for semi-norms as well. Although this procedure, strictly speaking, cancels the semi-norm property, we may modify the composite semi-norms in such a way that the resulting functions have an initial value in a natural way. Namely, we begin the recursive construction by using $L^1(\delta_0) \oplus_{p_1} L^{p_1}(\mu_1)$ as the first term, where δ_0 is the Dirac's delta probability measure concentrated at 0. The absolute value of the function at 0 then serves as the 'initial value of the semi-norm'.

Lemma 3.5. *Let $\rho(f) < \infty$. Suppose that there are compact subsets $C_i \subset [0, 1]$, $1 \leq i \leq n$, $\max C_i \leq \min C_{i+1}$ such that $p|_{C_i} \equiv p_i \in [1, \infty)$. Assume additionally*

that $f = 1_{\bigcup_i C_i} f$ and $p|_{[0,1] \setminus \bigcup_i C_i} \equiv 1$. Then the mapping $\varphi: [0, 1] \rightarrow \mathbb{R}$ given by $\varphi(t) = \rho(1_{[0,t]} f)$ is absolutely continuous and satisfies

$$\varphi(0) = 0, \quad \varphi'(t) = \frac{|f(t)|^{p(t)}}{p(t)} \varphi(t)^{1-p(t)} \quad \text{for a.e. } t \in [0, 1].$$

Proof. First we observe the analogous claim on an interval with a constant p by studying the following differential equation:

$$\varphi(a) = c, \quad \varphi'(t) = \frac{|f(t)|^p}{p} \varphi(t)^{1-p} \quad \text{for a.e. } t \in [a, b] \subset [0, 1].$$

We use the separability of the above differential equation and the absolute continuity of φ to obtain

$$(3.4) \quad \int_a^b p \varphi'(t) \varphi^{p-1} dt = \int_a^b \varphi^p(t) = \int_a^b |f(t)|^p dt.$$

Indeed, we see immediately that φ defined in the formulation of the Lemma is absolutely continuous in this special case. The above calculation considered in backward order shows also that in the constant p case φ arises as a solution to the above differential equation on that interval.

From this we obtain the analogous compact subset $C \subset [0, 1]$ case by passing to function of the type $1_C f$. It is clear that the resulting φ is again absolutely continuous and the derivative is

$$\varphi'(t) = 1_C(t) \frac{|f(t)|^p}{p} \varphi(t)^{1-p} = \frac{|1_C(t) f(t)|^p}{p} \varphi(t)^{1-p}.$$

This way we easily see that the semi-norm accumulation functions

$$(3.5) \quad t \mapsto \|1_{[0,t]} f\|_{(\dots(L^{p_1}(\mu_1) \oplus_{p_2} L^{p_2}(\mu_2)) \oplus_{p_3} \dots) \oplus_{p_n} L^{p_n}(\mu_n)}$$

can be seen as solutions to

$$\varphi(0) = 0, \quad \varphi'(t) = \frac{|f(t)|^{p(t)}}{p(t)} \varphi(t)^{1-p(t)} \quad \text{for a.e. } t \in [0, 1]$$

where $p(t) = p_i$ for $t \in C_i$ and $\varphi'(t) = 0$ for $t \in [0, 1] \setminus \bigcup_i C_i$. Indeed, for $x_i = \max C_i$ in (3.5) we obtain an ODE

$$\varphi(x_i) = \|1_{[0,x_i]} f\|, \quad \varphi'(t) = \frac{|f(t)|^{p(t)}}{p(t)} \varphi(t)^{1-p(t)} \quad \text{for a.e. } t \in C_{i+1}$$

by induction. Note that the sup in the limsup in (3.3) is actually attained in this simple case with $p(\cdot)$ essentially piecewise constant. \square

Given a measurable $p: [0, 1] \rightarrow [1, \infty)$, by Lusin's theorem there is for each $\varepsilon > 0$ a compact set $C \subset [0, 1]$ with $m([0, 1] \setminus C) < \varepsilon$ such that $p|_C$ is continuous, thus uniformly continuous and bounded. We denote by $\mathcal{C}_{p(\cdot)}$ the collection of such compact sets C partially ordered by inclusion.

Thus we can find a sequence of compact subsets $C_m \subset [0, 1]$ such as above with $m(C_m) \rightarrow 1$ and by taking finite unions of such sets we may assume that the sequence is increasing. In the following lemma we will study a technical tool, namely a functional given by

$$\|f\|_{\widetilde{L^{p(\cdot)}}} := \sup_{C \in \mathcal{C}_{p(\cdot)}} \rho(1_C f) = \lim_{C \nearrow \mathcal{C}_{p(\cdot)}} \rho(1_C f),$$

i.e. a supremum over compact subsets C such that $p|_C$ is continuous and the limit is taken over a net.

By using (3.5) it is easy to see that the above functional gives a complete norm in the space of functions $f \in L^0$ with $\|f\|_{\widetilde{L^{p(\cdot)}}} < \infty$. This space is denoted by $\widetilde{L^{p(\cdot)}}$.

Lemma 3.6. *Let $f \in L^{p(\cdot)}$. Then $\|f\|_{p(\cdot)} = \|f\|_{\widetilde{L^{p(\cdot)}}}$.*

Proof. Fix $p: [0, 1] \rightarrow [1, \infty)$ and $f \in L^{p(\cdot)}$. First, suppose that $f \in L^\infty$. Let (C_m) be a sequence of compact sets such as above.

We will later select the supports of μ_i 's inside these sets C_m . We can find a \leq -increasing sequence N_n such that

$$\begin{aligned} \phi_{n,m}(t) &:= \|1_{[0,t] \cap C_m} f\|_{N_n} \leq \phi_m := \|1_{[0,t] \cap C_m} f\|_{\widetilde{L^{p(\cdot)}}} \quad m \in \mathbb{N}, \\ \limsup_{n \rightarrow \infty} \phi_{n,m} &= \phi_m, \quad m \in \mathbb{N}. \end{aligned}$$

Indeed, here we apply the fact that ϕ_m have finite variation and the basic properties (e.g Markovian property) of the semi-norms $\|\cdot\|_N$. Alternatively, for each rational number $(q_k)_{k \in \mathbb{N}} = \mathbb{Q}$ pick a semi-norm N'_k with $\|1_{[0,q_k] \cap C_m} f\|_{N'_k} > \|1_{[0,q_k] \cap C_m} f\|_{\widetilde{L^{p(\cdot)}}} - 1/k$, $k \in \mathbb{N}$, then put $N_n = \bigvee_{k=1}^n N'_k$. It is easy to see that the functions ϕ_m are continuous, since $f \in L^\infty$ and $p(\cdot)$'s are effectively bounded. Moreover, the above sequence can be chosen in such a way that the diameters of the supports of the measures μ_i tend uniformly to 0 as $n \rightarrow \infty$. By a diagonal argument we may assume that the above lim sup is in fact lim.

By the uniform continuity of $p|_{C_m}$ we observe that the r_i and p_i appearing in the definition of N_n semi-norms satisfy $r_i - p_i \rightarrow 0$ uniformly on i as $n \rightarrow \infty$.

Recall Lemma 3.5. Since $f \in L^\infty$ and $p|_{C_m}$ are bounded we observe that

$$\phi'_{n,m}(t) \rightarrow \frac{|1_{C_m} f|^{p(t)}}{p(t)} \phi_m^{1-p(t)}(t)$$

in L^∞ on $C_m \cap \overline{\{t: \inf_{n,m} \phi_{n,m}(t) > \varepsilon\}}$ as $n \rightarrow \infty$. The same effect can be accomplished by using a joint initial value $x_0 = \varepsilon > 0$ and modifying the definitions of ϕ 's and N 's accordingly. This means that ϕ_{m,x_0} is absolutely continuous and ϕ'_{m,x_0} exists a.e. with

$$\phi_{m,x_0}(0) = x_0, \quad \phi'_{m,x_0}(t) = \frac{|1_{C_m} f|^{p(t)}}{p(t)} \phi_{m,x_0}^{1-p(t)}(t).$$

In letting $x_0 \searrow 0$ it is easy to see that the corresponding values $\phi_{m,x_0} \searrow \phi_m$ (recall the definition of the semi-norms, or (2.3)) and the monotone convergence theorem applies above. Therefore ϕ_m is an admissible solution, or, by shorthand,

$$(3.6) \quad \|1_{[0,t] \cap C_m} f\|_{p(\cdot)} = \phi_m(t).$$

Let $\phi := \limsup_{m \rightarrow \infty} \phi_m$. By the preceding observation and collecting all the subsets C_m we have that

$$\phi(T) \geq \int_0^T \frac{|f(t)|^{p(t)}}{p(t)} \varphi_f^{1-p(t)}(t) dt, \quad T \in [0, 1],$$

since $\phi(t) \leq \varphi_f(t)$ by (3.6). Now, by the selection of the functions $\phi_{n,m}$ we have that

$$\phi_{n,m}(T) \leq \sup_m \int_0^T \frac{|1_{C_m} f(t)|^{p(t)}}{p(t)} \phi_m^{1-p(t)}(t) dt = \sup_m \|1_{[0,T] \cap C_m} f\|_{p(\cdot)} = \|1_{[0,T]} f\|_{p(\cdot)}$$

where the first equality follows from (3.6) and the latter one from the absolute continuity of the solution φ_f . By combining the above inequalities we conclude that $\phi = \varphi_f$.

Next we will treat the case where $f \in L^{p(\cdot)}$ may be essentially unbounded. By Lusin's theorem, let $D_i \subset [0, 1]$ be a sequence of compact subsets such that both f and p are uniformly continuous on D_i , $D_i \subset C_i$ and $m(D_i) \rightarrow 1$ as $i \rightarrow \infty$. Let ψ_i be defined by

$$\psi_i(t) := \rho(1_{[0,t] \cap D_i} f) = \|1_{[0,t] \cap D_i} f\|_{p(\cdot)}$$

where we applied the conclusion obtained in the first part of the proof.

Let $\psi = \limsup_{i \rightarrow \infty} \psi_i$. By the construction of $\|\cdot\|_{p(\cdot)}$ and we immediately observe that $\psi \leq \varphi_f$. On the other hand,

$$\begin{aligned} \psi(T) &= \limsup_{i \rightarrow \infty} \int_0^T \frac{|1_{D_i}(t)f(t)|^{p(t)}}{p(t)} \psi_i^{1-p(t)} dt \geq \lim_{i \rightarrow \infty} \int_0^T \frac{|1_{D_i}(t)f(t)|^{p(t)}}{p(t)} \varphi_f^{1-p(t)} dt \\ &= \int_0^T \frac{|f(t)|^{p(t)}}{p(t)} \varphi_f^{1-p(t)} dt = \varphi_f(T). \end{aligned}$$

Thus $\psi = \varphi_f$.

Finally, note that

$$\|1_{[0,t] \cap C_m \cap D_i} f\|_{\widetilde{L^{p(\cdot)}}} \rightarrow \|1_{[0,t] \cap C_m} f\|_{\widetilde{L^{p(\cdot)}}}, \quad i \rightarrow \infty$$

since the analogous fact holds for all simple semi-norms $N \leq p(\cdot)$ separately. Indeed, this is due to the fact that the accumulation functions corresponding to the simple semi-norms are absolutely continuous.

We conclude that $\|f\|_{p(\cdot)} = \|f\|_{\widetilde{L^{p(\cdot)}}}$ for all $f \in L^{p(\cdot)}$. □

Remark 3.7. According to the previous result the $\|\cdot\|_{L^{p(\cdot)}}$ functional satisfies the triangle inequality (when applicable).

Remark 3.8. By inspecting the proof of the previous result we can conclude the following: For each $f \in L^{p(\cdot)}$ there is a sequence of simple functions (f_n) such that $f - f_n \rightarrow 0$ a.e. and $|f_n| \nearrow |f|$ a.e. and $\|f_n\|_{\widetilde{L^{p(\cdot)}}} \nearrow \|f\|_{L^{p(\cdot)}}$ as $n \rightarrow \infty$.

Remark 3.9. For $f \in L^\infty$ one may use standard form composite semi-norms in the above argument.

Remark 3.10. Inductively one sees that Hölder's inequality clearly holds on the standard form semi-norms. Thus the approximation argument in the proof of the previous result yields an alternative route to Hölder's inequality on $L^{p(\cdot)}$ spaces.

3.3. The essentially bounded exponent case. Let us take a look at the nice case where $\bar{p} := \text{ess sup}_t p(t) < \infty$ as it turns out that the corresponding spaces have less pathological properties.

We observed previously that $L^{p(\cdot)}$ classes need not have the ideal property in general. However, in the case $\bar{p} < \infty$ conditions $g \in L^{p(\cdot)}$, $f \leq g$ imply that also $f \in L^{p(\cdot)}$. This follows immediately from the following observation.

Proposition 3.11. *Suppose that $\bar{p} < \infty$, $g \in L^{p(\cdot)}$, $|f| \leq |g|$, and $0 < x_0 < 1$ is a given initial value. Then $f \in L^{p(\cdot)}$ and*

$$|\varphi'_{f,x_0}| \leq |\varphi'_{g,x_0}| \left(\frac{x_0}{\varphi_{g,x_0}(1)} \right)^{1-\bar{p}}.$$

Proof. Consider a simple semi-norm N applied to f, g and with the above initial value: $\phi(t) = \|1_{[0,t]}f\|_N$ and $\psi(t) = \|1_{[0,t]}g\|_N$. Clearly $\phi \leq \psi$. Denote by $p(\cdot)$ the corresponding piecewise constant exponent. According to Lemma 3.5 we may differentiate ϕ and ψ a.e. We obtain

$$\phi'(t) = \frac{|f(t)|^{p(t)}}{p(t)} \phi^{1-p(t)}(t)$$

and

$$\psi'(t) = \frac{|g(t)|^{p(t)}}{p(t)} \psi^{1-p(t)}(t)$$

so

$$\frac{\phi'(t)}{\psi'(t)} \leq \left(\frac{\phi(t)}{\psi(t)} \right)^{1-p(t)} \leq \left(\frac{\phi(t)}{\psi(t)} \right)^{1-\bar{p}} \leq \left(\frac{x_0}{\psi(1)} \right)^{1-\bar{p}}.$$

The existence of the solution for f follows from subsequent Theorem 3.13. \square

In particular we remain within the class if we restrict supports to a measurable subset. Therefore it is possible in principle to formulate kind of negative-initial-value solutions, simply by setting the function $f \in L^{p(\cdot)}$ to zero in a suitable initial segment.

Proposition 3.12. *Let $\bar{p} < \infty$, $f \in L^{p(\cdot)}$ and $A_n \subset [0, 1]$ a sequence of measurable subsets such that $m(A_n) \rightarrow 0$ as $n \rightarrow \infty$. Then $\|1_{A_n}f\|_{p(\cdot)} \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. Fix $\varepsilon > 0$. We claim that given initial value $x_0 = \varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$(3.7) \quad \varphi_{1_{A_n}f, x_0}(1) < 2\varepsilon, \quad n \geq n_0.$$

This clearly suffices for the statement of the lemma.

The absolute continuity of the solution φ_{f, x_0} implies that

$$\int_{A_n} \varphi'_{f, x_0}(t) dt \rightarrow 0, \quad n \rightarrow \infty.$$

Then this observation together with the previous remark yields (3.7). \square

Then $L^\infty \subset L^{p(\cdot)}$ is dense by the triangle inequality. For a general measurable exponent $p(\cdot)$ we define a natural Banach subspace

$$L_0^{p(\cdot)} = \overline{\bigcup_{n \in \mathbb{N}} \left\{ 1_{p(t) \leq n} f : f \in \widetilde{L^{p(\cdot)}} \right\}} \subset \widetilde{L^{p(\cdot)}}.$$

Theorem 3.13. *For a general measurable exponent $p(\cdot)$ the above Banach space $L_0^{p(\cdot)} \subset L^{p(\cdot)}$. In particular, in the case $\bar{p} < \infty$, $L^{p(\cdot)} = \widetilde{L^{p(\cdot)}}$ and consequently is a Banach space.*

Proof. First we will verify the latter part of the statement, so assume that $\bar{p} < \infty$. Let $f \in \widetilde{L^{p(\cdot)}}$. Similarly as above, let $D_i \subset [0, 1]$, $i \in \mathbb{N}$, be measurable compact subsets such that both $f|_{D_i}$ and $p|_{D_i}$ are continuous and $m(D_i) \rightarrow 1$ as $i \rightarrow \infty$. Put

$$\psi_i(t) := \|1_{[0,t] \cap D_i} f\|_{\widetilde{L^{p(\cdot)}}}$$

where we consider the functions with a joint initial value $0 < x_0 < 1$. Note that

$$\psi_i(t) \nearrow \psi(t) := \|1_{[0,t]} f\|_{\widetilde{L^{p(\cdot)}}}$$

for $t \in [0, 1]$ as $i \rightarrow \infty$ by the absolute continuity of simple semi-norms. Recalling the proof of Lemma 3.6 we obtain that ψ_i are absolutely continuous and

$$\psi'_i = \frac{|1_{D_i} f|^{p(t)}}{p(t)} \psi_i^{1-p(t)}(t)$$

a.e.

Using the positivity of the initial value x_0 and $\bar{p} < \infty$, we obtain by studying the derivatives ψ'_i that

$$(3.8) \quad \psi(1)^{1-\bar{p}} \int_r^t \frac{|f(s)|^{p(s)}}{p(s)} ds \leq \psi(t) - \psi(r) \leq x_0^{1-\bar{p}} \int_r^t \frac{|f(s)|^{p(s)}}{p(s)} ds.$$

We conclude that ψ is absolutely continuous. According to Dini's theorem $\psi_i \rightarrow \psi$ converges uniformly on $[0, 1]$. Moreover, by using again the positivity of the initial value and $\bar{p} < \infty$ we get $\psi_i^{1-p(t)}(t) \rightarrow \psi^{1-p(t)}(t)$ in L^∞ -norm as $i \rightarrow \infty$. Thus

$$\psi'_i \rightarrow \frac{|f|^{p(t)}}{p(t)} \psi^{1-p(t)}$$

in L^1 and hence

$$\psi(T) = x_0 + \int_0^T \frac{|f|^{p(t)}}{p(t)} \psi^{1-p(t)}, \quad T \in [0, 1].$$

This shows that ψ is a solution witnessing the fact that $f \in L^{p(\cdot)}$, since x_0 was arbitrary.

To verify the first part of statement, fix $f \in L_0^{p(\cdot)}$ and we aim to show that $f \in L^{p(\cdot)}$, i.e. that there is a solution φ_f . Let $f_n = P_n f$ for $n \in \mathbb{N}$. Clearly $f_n \rightarrow f$ a.e. as $i \rightarrow \infty$. Similarly as above, it follows from the triangle inequality that $\varphi_{f_n} \rightarrow \varphi$ uniformly for a suitable ϕ . We consider the solutions with a joint positive initial value $x_0 > 0$.

For each $k \in \mathbb{N}$ and $\varepsilon > 0$ there exist by Egorov's theorem a set $D \subset \{t \in [0, 1] : p(t) \leq k\}$ such that $m(\{t \in [0, 1] : p(t) \leq k\} \setminus D) < \varepsilon$ and

$$\frac{|f_n(t)|^{p(t)}}{p(t)} \varphi_{f_n}^{1-p(t)}(t) \rightarrow \frac{|f(t)|^{p(t)}}{p(t)} \phi^{1-p(t)}(t)$$

uniformly on D as $n \rightarrow \infty$. Thus

$$\int_D \frac{|f_n(t)|^{p(t)}}{p(t)} \varphi_{f_n}^{1-p(t)}(t) dt \rightarrow \int_D \frac{|f(t)|^{p(t)}}{p(t)} \phi^{1-p(t)}(t) dt.$$

Since ε was arbitrary we get by Proposition 3.12 that

$$\int_{p(t) \leq k} \frac{|f_n(t)|^{p(t)}}{p(t)} \varphi_{f_n}^{1-p(t)}(t) dt \rightarrow \int_{p(t) \leq k} \frac{|f(t)|^{p(t)}}{p(t)} \phi^{1-p(t)}(t) dt$$

for each $k \in \mathbb{N}$. Since $\|f - P_n f\|_{\widetilde{L^{p(\cdot)}}} \rightarrow 0$, the corresponding minorating solutions tend to 0, thus we see that

$$\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{p(t) \geq k} \frac{|f(t)|^{p(t)}}{p(t)} \varphi_{f_n}^{1-p(t)}(t) dt = 0.$$

It follows that

$$\int_0^T \frac{|f_n(t)|^{p(t)}}{p(t)} \varphi_{f_n}^{1-p(t)}(t) dt \rightarrow \int_0^T \frac{|f(t)|^{p(t)}}{p(t)} \phi^{1-p(t)}(t) dt, \quad T \in [0, 1].$$

Taking into account that $\varphi_{f_n} \rightarrow \phi$ uniformly, we see that

$$\phi(T) = x_0 + \int_0^T \frac{|f(t)|^{p(t)}}{p(t)} \phi^{1-p(t)}(t) dt, \quad T \in [0, 1].$$

□

The above argument (recall (3.8)) yields the following fact.

Proposition 3.14. *If $\bar{p} < \infty$, then $f \in L^0$ is in $L^{p(\cdot)}$ if and only if*

$$\int_0^1 \frac{|f(t)|^{p(t)}}{p(t)} dt < \infty.$$

□

4. DUALITY

Given a function $g: [0, 1] \rightarrow \mathbb{R}$ with finite variation, let us denote a special ‘variation norm’ as follows:

$$\bigvee_{p(\cdot)^*} m_g = \bigvee_{p(\cdot)^*} g = \sup \left\{ \int_0^1 f dm_g : f \in C[0, 1], \|f\|_{p(\cdot)} \leq 1 \right\}.$$

Here m_g is the Lebesgue-Stieltjes measure induced by g . For a continuously differentiable g the notable special cases are

$$\bigvee_{(p \equiv 1)^*} g = \text{Lip}(g),$$

the best Lipschitz constant of g , and the usual total variation

$$\bigvee_{(p \equiv \infty)^*} g = \bigvee g.$$

The above notion is applied somewhat tautologically in the following result.

Theorem 4.1. *Let*

$$X = \overline{C[0, 1]} \subset L^{p(\cdot)}.$$

Then the dual space X^ elements are Lebesgue-Stieltjes measures m_g with finite $\bigvee_{p(\cdot)^*} m_g$ variation. The dual space is endowed with the norm*

$$\|m_g\|_{X^*} = \bigvee_{p(\cdot)^*} m_g$$

and the duality is given by

$$\langle F, x \rangle = \int_0^1 x(t) dm_g(t), \quad x \in X,$$

the Lebesgue integral with Lebesgue-Stieltjes measure m_g induced by $g(t) = F(1_{[0,t]})$ for $F \in X^$.*

Proof. Let us begin by studying continuous linear functionals F on the normed space $C[0, 1] \subset L^{p(\cdot)}$. Since $\|\cdot\|_{p(\cdot)} \leq e\|f\|_\infty$ we obtain that each $F \in (C[0, 1], \|\cdot\|_{p(\cdot)})^*$ is also bounded with respect to the norm $\|\cdot\|_\infty$. Thus $F \in (C[0, 1], \|\cdot\|_{p(\cdot)})^* \subset (C[0, 1], \|\cdot\|_\infty)^*$ with the usual duality

$$\langle F, f \rangle = \int f(t) dg(t), \quad g(t) = F(1_{[0,t]})$$

and

$$\bigvee g \leq e \|F\|_{(C[0,1], \|\cdot\|_{L^{p(\cdot)}})^*}.$$

To allow for integrating non-continuous functions without any difficulty we will integrate in the more general Lebesgue-Stieltjes sense in taking duality. Thus, let m_g be the Lebesgue-Stieltjes measure induced by g . We note that F is a continuous linear functional on $(C[0,1], \|\cdot\|_{L^{p(\cdot)}})$, the above duality holds, if and only if

$$\langle F, f \rangle = \int f \, dm_g, \quad f \in C[0,1],$$

$g(t) = F(1_{[0,t]})$. Here $\|F\|_{X^*} = \bigvee_{p(\cdot)^*} m_g$ by the definition of the special variation.

Let us verify that the above integral representation extends continuously to the closure $\overline{C[0,1]} \subset L^{p(\cdot)}$ for each $F \in X^*$. Fix $x \in \overline{C[0,1]}$. Pick $(x_n) \subset C[0,1]$ such that $\|x_n - x\|_{L^{p(\cdot)}} \rightarrow 0$ as $n \rightarrow \infty$. Since (x_n) is Cauchy, we can extract a subsequence (n_j) such that $x_{n_1} + \sum_j x_{n_{j+1}} - x_{n_j} = x$ unconditionally in the $L^{p(\cdot)}$ -norm and $\sum_j \|x_{n_{j+1}} - x_{n_j}\|_{p(\cdot)} < \infty$. It follows from the definition of $\bigvee_{p(\cdot)^*} m_g$ that then

$$(4.1) \quad \sum_j \left| \int (x_{n_{j+1}} - x_{n_j})(t) \, dm_g(t) \right| < \infty.$$

By passing to a further subsequence and modifying all the functions x_{n_j} and x in a m_g -null set we may assume that $x_{n_1}(t) + \sum_j (x_{n_{j+1}} - x_{n_j})(t) = x(t)$ for every t .

Consider the Banach space $L^1(m_g)$. We obtain from (4.1) that $x_{n_j} \rightarrow y$ in the norm $\|\cdot\|_{L^1(m_g)}$. Also, we observe that $y(t) = x(t)$ for m_g -a.e. t by convergence in m_g -measure considerations. We conclude that

$$\int x_{n_j}(t) \, dm_g(t) \rightarrow \int x(t) \, dm_g(t), \quad j \rightarrow \infty.$$

It is easy to see that the above convergence does not depend on the particular selection of the approximating Cauchy sequence of continuous functions. \square

Let $p: [0,1] \rightarrow (1, \infty)$ be a measurable function. Let $X_n \subset L^{p(\cdot)}$, $n \in \mathbb{N}$, be the images of the contractive projections $P_n: L^{p(\cdot)} \rightarrow X_n$, $(P_n f)(t) = 1_{1+1/n \leq p(t) \leq n}(t) f(t)$. Recall that we denote by

$$L_0^{p(\cdot)} = \overline{\bigcup_n X_n} \subset L^{p(\cdot)}.$$

Actually,

$$L_0^{p(\cdot)} = \overline{\bigcup_n \{1_{p(t) \leq n} f: f \in L^{p(\cdot)}\}} := X.$$

This is seen as follows: we claim that for each $f \in X$ we have

$$\|f - P_n f\| + \|P_n f\| \rightarrow \|f\|, \quad n \rightarrow \infty.$$

For fixed initial value $a > 0$ and $\varphi_{f-P_n f}(0) = \varphi_{P_n f}(0) = \varphi_f(0) = a$ the analogous statement follows easily since $\varphi^{1-p(t)} \rightarrow 1$ as $p(t) \searrow 1$. By the absolute continuity of the solutions we obtain that $\varphi_{P_n f}(1) \rightarrow \varphi_f(1)$ as $n \rightarrow \infty$ for any given initial value $a > 0$. Thus $\varphi_{f-P_n f}(1) \rightarrow a$ as $n \rightarrow \infty$ for any initial value $a > 0$. By a diagonal argument we find a sequence $a_n \searrow 0$ of initial values such that $\varphi_{f-P_n f, a_n}(1) \rightarrow 0$ as $n \rightarrow \infty$. Since the solutions are non-decreasing with respect to their initial values, we obtain that $\|f - P_n f\| \rightarrow 0$ as $n \rightarrow \infty$. \square

Theorem 4.2. *If $1 < \text{ess inf}_t p(t) \leq \text{ess sup}_t p(t) < \infty$ then for each $F \in (L^{p(\cdot)})^*$ there is $f \in L^{p^*(\cdot)}$ such that*

$$\langle F, x \rangle = \int x(t)f(t) \, dm(t), \quad \text{for all } x \in L^{p(\cdot)}$$

and the above duality induces an isomorphism $(L^{p(\cdot)})^ \rightarrow L^{p^*(\cdot)}$. Moreover, $(L_0^{p(\cdot)})^*$ is isomorphic to $\widetilde{L^{p^*(\cdot)}}$ with the above duality for a general $p: [0, 1] \rightarrow (1, \infty)$.*

Proof. It follows from an easy adaptation of Hölder's inequality that $\widetilde{L^{p^*(\cdot)}} \subset (L^{p(\cdot)})^*$ in the sense that

$$|F(x)| = \left| \int x f \, dm \right| \leq \|x\|_{p(\cdot)} \|f\|_{p^*(\cdot)}, \quad x \in L^{p(\cdot)}$$

whenever $f \in \widetilde{L^{p^*(\cdot)}}$ is regarded as a function and F is in the subspace with the usual identification (4.2).

Let us begin by verifying the statement in the reflexive case, i.e. $\text{ess inf}_t p(t) > 1$ and $\text{ess sup}_t p(t) < \infty$ (see Theorem 5.2), so that we are actually studying a space X_n for a given n . Let $F \in (L^{p(\cdot)})^*$. By modifying the standard proof (see e.g. [8, Prop. 2.17]) of the statement in the usual constant exponent case we obtain that there is f such that

$$(4.2) \quad \langle F, x \rangle = \int x(t)f(t) \, dm(t)$$

holds for every measurable bounded x .

Note that $\|F\|_{X^*} \leq \|f\|_{p^*(\cdot)}$. By applying Lebesgue's monotone convergence theorem we observe that we may approximate

$$\int x_n f \, dm \rightarrow \int x f \, dm$$

by simple functions x_n and that (4.2) holds for all $x \in L^{p(\cdot)}$.

Let $C = 1 + ae$ as in Proposition 3.2. We claim that $C\|F\|_{(L^{p(\cdot)})^*} \geq \|f\|_{p^*(\cdot)}$ which together with the above estimate suffices for the equivalence of the norms. Suppose that $\|f\|_{p^*(\cdot)} = 1$ and let N be a standard form semi-norm with $p^*(t) \geq p_i^* > 1$ for $t \in \text{supp } \mu_i$ and $\|f\|_N \geq 1 - \varepsilon$. Let N_* be the (pre)dual standard form semi-norm of N , that is, with p_i in place of p_i^* . The duality of such semi-norms is clear. Note that $p(t) \leq p_i < \infty$. Thus we can find a simple function g supported in the support of N_* and such that $\int_0^1 g f \geq 1 - 2\varepsilon$ and $\|g\|_{N_*} = 1$. By the approximate monotonicity of the norms we get that $\|g\|_{p(\cdot)} \leq C$. This gives that $C\|F\|_{(L^{p(\cdot)})^*} \geq \|f\|_{p^*(\cdot)}$.

Next we treat the non-reflexive case. As pointed above, it follows from Hölder's inequality that $\widetilde{L^{p^*(\cdot)}} \subset (L_0^{p(\cdot)})^*$ and $\|F\|_{(L_0^{p(\cdot)})^*} \leq \|f\|_{\widetilde{L^{p^*(\cdot)}}}$. Pick $f \in \widetilde{L^{p^*(\cdot)}}$. Restrict the corresponding F to the subspace $\bigcup_n X_n$. This does not change the operator norm, since the subspace is dense. It is easy to see that $\|P_n f\|_{L^{p^*(\cdot)}} \rightarrow \|f\|_{\widetilde{L^{p^*(\cdot)}}}$ as $n \rightarrow \infty$. Hence, by using the observations of the reflexive case, we may pick for each $\varepsilon > 0$ such n and $x \in X_n$, $\|x\|_{L^{p(\cdot)}} = C$, that $|(P_n^* f)(x)| > 1 - \varepsilon$. Thus we observe that $\widetilde{L^{p^*(\cdot)}} \subset (L_0^{p(\cdot)})^*$ is an isomorphic subspace. Pick $F \in (L_0^{p(\cdot)})^*$. Restrict F to $\bigcup_n X_n$. Since the projections P_n commute, this produces a natural candidate for the representation, namely $f = \lim_n P_n^* F$, the limit taken point-wise a.e. Since each $P_n F \in L^{p^*(\cdot)}$ and $\|P_n F\|_{L^{p^*(\cdot)}} \leq C\|F\|_{(L_0^{p(\cdot)})^*}$ we obtain that

$f \in \widetilde{L^{p^*(\cdot)}}$, although the above limit does *not*, a priori, exist in the $\widetilde{L^{p^*(\cdot)}}$ norm. Let us verify that f presents F . Pick $x \in L_0^{p(\cdot)}$. Then

$$F(x) - \int x(t) f(t) dm(t) = F(x - P_n x) + F(P_n x) - \left(\int (x - P_n x) f dm + \int P_n x f dm \right).$$

Here $F(x - P_n x) \rightarrow 0$ by the continuity of the functional and $\int (x - P_n x) f dm \rightarrow 0$ by Hölder's inequality. On the other hand,

$$F(P_n x) = (P_n^* f)(x) = \int 1_{1+1/n \leq p(t) \leq n} f(t) x(t) dm(t) = \int (P_n x)(t) f(t) dm(t).$$

Thus $F(x) = \int x(t) f(t) dm(t)$ for all $x \in L_0^{p(\cdot)}$. This concludes the proof. \square

4.1. Duality and ODEs. The previous result gives an anticipated duality for $L^{p(\cdot)}$ spaces, but the dual norms and p^* -norms do not coincide isometrically. Next we will study this issue and return to the considerations involving Hölder's inequality. Let us denote by J the 'duality map' $J: L^{p(\cdot)} \rightarrow L^0$,

$$(4.3) \quad J(x)[t] = \text{sign}(x(t)) |x(t)|^{\frac{p(t)}{p^*(t)}}, \quad p, p^* \in L^0, \quad \frac{1}{p(t)} + \frac{1}{p^*(t)} = 1.$$

Next we will perform some calculations. We will use rules on ∂^* consistent with the definition of the solutions φ . Going back to (3.2), we are interested here in the equality which corresponds to the duality case. According to Young's inequality and its proof it is known that the equality can only hold if

$$(4.4) \quad \varphi_x^{p(t)} = \varphi_{x^*}^{p^*(t)} \quad \text{for a.e. } t.$$

Let us calculate

$$\partial^* \varphi_x^{p(t)} = p(t) (\partial^* \varphi_x) \varphi_x^{p(t)-1} = p(t) \frac{|x(t)|^{p(t)}}{p(t)} \varphi_x^{1-p(t)} \varphi_x^{p(t)-1} = |x(t)|^{p(t)}$$

and similarly

$$\partial^* \varphi_{x^*}^{p^*(t)} = |x^*(t)|^{p^*(t)} = (|x(t)|^{\frac{p(t)}{p^*(t)}})^{p^*(t)} = |x(t)|^{p(t)}.$$

Therefore we conclude that

$$(4.5) \quad \varphi_x^{p(t)} = \varphi_{x^*}^{p^*(t)} \quad \text{and} \quad |x(t)|^{p(t)} = |x^*(t)|^{p^*(t)} \quad \text{a.e.}$$

if the solutions φ_x and φ_{x^*} specified in (1.6) exist and satisfy

$$(4.6) \quad (\varphi_x^{p(\cdot)})(T) = \int_0^T \partial^* \varphi_{x^*}^{p^*(t)} dt \quad \text{and} \quad (\varphi_{x^*}^{p^*(\cdot)})(T) = \int_0^T \partial^* \varphi_x^{p(t)} dt.$$

The latter requirement imposes a strong continuity condition on the exponent.

Note that in the case where $1 < \text{ess inf}_t p(t) \leq \text{ess sup}_t p(t) < \infty$ and the solutions φ have a positive initial value we have that

$$(\log \varphi_{x^*})'(t) = \frac{\varphi_{x^*}'(t)}{\varphi_{x^*}(t)} \leq C \varphi_x'(t) \quad \text{a.e.}$$

for a suitable constant $C > 0$. This inequality together with a suitable approximation of x and x^* from below by increasing supports, as in Lemma 3.6, gives that J maps $L^{p(\cdot)} \rightarrow L^{p^*(\cdot)}$ for $p(\cdot)$ such as above.

Let us study the validity of the following statement:

$$\int_0^T x(t) x^*(t) dt = \varphi_x(T) \varphi_{x^*}(T), \quad 0 \leq T \leq 1.$$

Recall that $\frac{p(t)}{p^*(t)} = p(t) - 1$. By using that $\varphi_x^{p(t)} = \varphi_{x^*}^{p^*(t)}$, we get $\varphi_{x^*}(t) = \varphi_x^{\frac{p(t)}{p^*(t)}}$ and thus

$$(4.7) \quad \varphi_x(t)\varphi_{x^*}(t) = \varphi_x^{1 + \frac{p(t)}{p^*(t)}}.$$

We will differentiate the right hand side to obtain the claim as follows:

$$\begin{aligned} (\varphi_x \varphi_{x^*})' &= \partial^*(\varphi_x \varphi_{x^*}) = \partial^* \varphi_x^{1 + \frac{p(t)}{p^*(t)}} = \left(1 + \frac{p(t)}{p^*(t)}\right) \varphi_x' \varphi_x^{\frac{p(t)}{p^*(t)}} \\ &= (p(t)) \frac{|x(t)|^{p(t)}}{p(t)} \varphi_x^{1-p(t)} \varphi_x^{p(t)-1} = |x(t)|^{p(t)} = |x(t)||x(t)|^{p(t)-1} = |x(t)||x^*(t)|. \end{aligned}$$

Proposition 4.3. *Suppose $1 < \text{ess inf}_t p(t) \leq \text{ess sup}_t p(t) < \infty$ and the exponentiated solutions $\varphi_x^{p(t)}$ and $\varphi_{x^*}^{p^*(t)}$ satisfy the above continuity condition (4.6). Then the duality mapping*

$$J(x)[t] = \text{sign}(x(t))|x(t)|^{\frac{p(t)}{p^*(t)}}$$

satisfies

$$\langle J(x), x \rangle = \|x\|_{p(\cdot)} \|J(x)\|_{p^*(\cdot)}.$$

Moreover, the following diagram commutes:

$$\begin{array}{ccc} x & \xrightarrow{J} & x^* \\ \downarrow \varphi \cdot & & \downarrow \varphi \cdot \\ \varphi_x & \xrightarrow{J} & \varphi_{x^*}. \end{array}$$

□

Interestingly, under suitable higher regularity conditions, comparable to (4.6) (e.g. constant p case), not only the functions x and $x^* = J(x)$ are dual in the sense of the usual pairing but also the solutions are dual to each other:

$$\langle \varphi_{x^*}, \varphi_x \rangle = \|\varphi_x\|_p \|\varphi_{x^*}\|_{p^*}.$$

Using (4.3) and (4.4) on φ_x' and φ_{x^*}' , written as in the basic ODE (1.6), provides us with the following identity

$$(4.8) \quad \varphi_{x^*}'(t) = \varphi_x'(t) \frac{p(t)}{p^*(t)} \frac{\varphi_{x^*}(t)}{\varphi_x(t)}.$$

It is instructive to observe how $p \equiv p^* \equiv 2$ (first with a joint positive initial value) results in the self-dual situation, $\varphi_x = \varphi_{x^*}$. The duality of the differential equations in (4.8) yields also intrinsic information on the $L^{p(\cdot)}$ type ODEs, see Theorem 5.4.

5. EXTENSIONS OF THE DEFINITION OF THE SPACES

5.1. Change of variable. Let us consider an equivalent measure $\mu \sim m$ on the unit interval and $\frac{d\mu}{dm}$ with

$$\mu(A) = \int_A \frac{d\mu}{dm}(t) dm(t)$$

for all Borel sets A . The above Radon-Nikodym derivative need not be integrable. Going back to the heuristical derivation of the norm-determining ODE and repeating the considerations with $L^p(\mu)$ in place of L^p under the assumption that $\frac{d\mu}{dm}(t)$ is a continuous function, we arrive at the following ODE:

$$(5.1) \quad \varphi(0) = 0^+, \quad \varphi'(t) = \frac{d\mu}{dm}(t) \frac{|f(t)|^{p(t)}}{p(t)} \varphi(t)^{1-p(t)} \quad \text{for } m\text{-a.e. } t \in [0, 1].$$

Similarly as above we define a space of functions together with a norm (for a general $\mu \sim m$) and we denote this space by $L^{p(\cdot)}(\mu)$. This can be regarded as a ‘weighted’ $L^{p(\cdot)}$ space and provide examples of the wider class of spaces introduced in Section 2. Recall that $L^p([0, 1])$ and $L^p(\mathbb{R})$ are isometric; the same reasoning extends to our setting.

Proposition 5.1. *Let $p: [0, 1] \rightarrow [1, \infty)$ be measurable and $\mu \sim m$. Then the mapping*

$$T: f(t) \mapsto \left(\frac{d\mu}{dm}(t) \right)^{-\frac{1}{p(t)}} f(t)$$

is a surjective linear isometry $L^{p(\cdot)} \rightarrow L^{p(\cdot)}(\mu)$.

Proof. Clearly the mapping is linear. Isometry follows by calculation:

$$\begin{aligned} \varphi'_{\mu, T(f)}(t) &= \frac{d\mu}{dm}(t) \frac{\left| \left(\frac{d\mu}{dm}(t) \right)^{-\frac{1}{p(t)}} f(t) \right|^{p(t)}}{p(t)} \varphi_{\mu, T(f)}(t)^{1-p(t)} = \frac{|f(t)|^{p(t)}}{p(t)} \varphi_{\mu, T(f)}(t)^{1-p(t)} \\ &= \frac{|f(t)|^{p(t)}}{p(t)} \varphi_{m, f}(t)^{1-p(t)} = \varphi'_{m, f}(t). \end{aligned}$$

Indeed, a moment’s reflection involving a joint positive initial value justifies the fact $\varphi_{\mu, T(f)} = \varphi_{m, f}$. Surjectivity follows by observing that

$$f(t) \mapsto \left(\frac{d\mu}{dm}(t) \right)^{\frac{1}{p(t)}} f(t)$$

defines the inverse of the operator. □

5.2. Applications of changing density.

Theorem 5.2. *Let $p: [0, 1] \rightarrow (1, \infty)$ be measurable. The following conditions are equivalent:*

- (1) $L^{p(\cdot)}$ is uniformly convex and uniformly smooth.
- (2) $L^{p(\cdot)}$ is reflexive.
- (3) $L_0^{p(\cdot)}$ contains neither ℓ^1 , nor c_0 almost isometrically.
- (4) $\text{ess inf}_t p(t) > 1$ and $\text{ess sup}_t p(t) < \infty$.

Proof. The implications (1) \implies (2) \implies (3) are clear.

The direction (3) \implies (4). Suppose that $\text{ess inf}_t p(t) = 1$. We will show that then $L_0^{p(\cdot)}$ contains an isomorphic copy of ℓ^1 for any isomorphism constant $C > 1$.

By the compactness of the unit interval we can find a point t_0 such that

$$\text{ess inf}_t 1_{(t_0-\varepsilon, t_0+\varepsilon)}(t) p(t) = 1 \quad \text{for each } \varepsilon > 0.$$

Indeed, assume that this is not the case and consider a suitable open cover of open intervals $(t_0 - \varepsilon, t_0 + \varepsilon)$, so that there is a finite subcover contradicting $\text{ess inf}_t p(t) =$

1. Therefore we may extract a sequence (A_n) of measurable subsets of the unit interval with positive measure such that the following conditions hold:

- (1) $\sup p|_{A_n} \searrow 1$ as $n \rightarrow \infty$.
- (2) Either $\max A_n < \min A_{n+1}$ for all n or $\max A_n > \min A_{n+1}$ for all n .

Fix a rapidly decreasing sequence of exponents $p_i \searrow 1$ such that

$$(5.2) \quad \prod_i \|I: \ell^{p_i}(2) \rightarrow \ell^1(2)\| < 1 + \varepsilon.$$

We can find a strictly increasing sequence (n_i) such that $p_i \geq p|_{A_{n_i}}$ for each $i \in \mathbb{N}$.

Let μ be an equivalent measure on the unit interval such that $\mu(A_{n_i}) = 1$ for $i \in \mathbb{N}$. In proving the claim it suffices study $L^{p(\cdot)}(\mu)$ in place of $L^{p(\cdot)}$, since these spaces are isometric. Put $\tilde{p}(t) = \max(1, \sum_i p_i 1_{A_{n_i}}(t))$.

Define a mapping $T: \ell^1 \rightarrow L^{p(\cdot)}(\mu)$ by putting

$$T((x_i)) = \sum_i x_i 1_{A_{n_i}}$$

where the sum is defined point-wise a.e.

We follow the arguments in [30] involving sequence space semi-norms arising as follows. For $(x_n) \in \ell^0$ we put

$$(\dots(|x_1| \boxplus_{p_1} |x_2|) \boxplus_{p_2} |x_3|) \boxplus_{p_3} \dots \boxplus_{p_{n-1}} |x_n|) \boxplus_{p_n} |x_{n+1}|,$$

in case (A_n) is increasing, or the analogous left-handed version if (A_n) is decreasing:

$$|x_1| \boxplus_{p_1} (|x_2| \boxplus_{p_2} (|x_3| \boxplus_{p_3} \dots \boxplus_{p_{n-2}} (|x_{p_{n-1}}| \boxplus_{p_{n-1}} (|x_n| \boxplus_{p_n} |x_{n+1}|) \dots)),$$

we observe that one may control inductively the difference of norms when one changes the values of the exponents p_i by using (5.2). That is,

$$\left\| \sum_i x_i 1_{A_{n_i}} \right\|_{L^{p(\cdot)}(\mu)} \geq \frac{1}{1 + \varepsilon} \sum_i \|x_i 1_{A_{n_i}}\|_{L^{p(\cdot)}(\mu)}.$$

Thus, $\|T^{-1}: T(\ell^1) \rightarrow \ell^1\| \leq 1 + \varepsilon$.

Similarly, by passing to subsequences of (A_n) multiple times we obtain that

$$\begin{aligned} \sum_i \|x_i 1_{A_{n_i}}\|_{L^1(\mu)} &= \left\| \sum_i x_i 1_{A_{n_i}} \right\|_{L^1(\mu)} \leq (1 + \varepsilon) \left\| \sum_i x_i 1_{A_{n_i}} \right\|_{L^{p(\cdot)}(\mu)} \\ &\leq (1 + 2\varepsilon) \left\| \sum_i x_i 1_{A_{n_i}} \right\|_{L^{\tilde{p}(\cdot)}(\mu)} \leq (1 + 3\varepsilon) \|(x_n)\|_{\ell^{\tilde{p}}} \\ &\leq (1 + 4\varepsilon) \|(x_n)\|_{\ell^1} = (1 + 4\varepsilon) \sum_i \|x_i 1_{A_{n_i}}\|_{L^1(\mu)}. \end{aligned}$$

Indeed, analyzing the $L^{p(\cdot)}$ -differential equation shows that for a constant function the values of the derivative uniformly approximate $|f(t)|$ as $p(t) \searrow 1$. Thus $\|T\| \leq 1 + \varepsilon$. This shows that the space contains ℓ^1 almost isometrically.

Next, assume that $\text{ess sup}_t p(t) = \infty$. We will show that $L_0^{p(\cdot)}$ contains c_0 almost isometrically. We may again without loss of generality make some assumptions about the equivalent measure, namely, that $\mu([0, 1]) = 1$ and

$$\mu(\{t \in [0, 1]: p(t) > r\})^{\frac{1}{r}} \rightarrow 1, \quad r \rightarrow \infty.$$

We will partition each set $\{t \in [0, 1] : n < p(t) \leq n+1\}$ to measurable subsets of equal μ -measure, call them $A_{n,0}^{(1)}$ and $A_{n,1}^{(1)}$. (Possibly both the subsets have measure 0.) Divide $A_{n,1}^{(1)}$ again to two subsets of equal measure, $A_{n,0}^{(2)}$ and $A_{n,1}^{(2)}$. We proceed recursively in this manner to construct sets $A_{n,\theta}^{(k)}$, $k, n \in \mathbb{N}$, $\theta \in \{0, 1\}$. Let $A_j^{(k)} = \bigcup_{n \geq j} A_{n,0}^{(k)}$. Observe that

$$\mu(A_j^{(k)}) = 2^{-k} \mu(\{t \in [0, 1] : p(t) > j\}), \quad k, j \in \mathbb{N}.$$

Note that

$$(5.3) \quad \lim_{j \rightarrow \infty} \mu(A_j^{(k)})^{\frac{1}{j}} = \lim_{j \rightarrow \infty} (2^{-k})^{\frac{1}{j}} \mu(\{t \in [0, 1] : p(t) > j\})^{\frac{1}{j}} = 1, \quad k \in \mathbb{N}.$$

Assume first that $1_{A_j^{(n)}} \in L^{p(\cdot)}(\mu)$, although this is not necessarily the case (see remarks in Section 3). Define an operator $T : c_{00} \rightarrow L^{p(\cdot)}(\mu)$ by

$$T((x_n)) = \sum_n x_n 1_{A_j^{(n)}}$$

defined point-wise a.e. Clearly $\|T\| \leq \|1\|_{\widetilde{L^{p(\cdot)}(\mu)}}$. In fact, by choosing a large enough j we get that $\|T\| \leq 1 + \varepsilon$. Indeed, observe that if $\varphi(t) \geq 1$ then $\frac{1}{j} \varphi^{1-j}(t)$ becomes small for a large j . Thus

$$(1 + \varepsilon) \max_n |x_n| \geq \|T((x_n))\|_{L^{p(\cdot)}(\mu)} \geq \max_n \|T(x_n e_n)\|_{L^{p(\cdot)}(\mu)}.$$

Here $(T(e_n))_n \subset L^{p(\cdot)}(\mu)$ is a 1-unconditional sequence. To show the claim it is required to check that

$$\|T(e_n)\|_{L^{p(\cdot)}(\mu)} \geq 1 - \varepsilon, \quad n \in \mathbb{N}.$$

This is seen as follows, first observe that

$$\|1_{A_0^{(n)}}\|_{L^{p(\cdot)}(\mu)} \geq \|1_{A_j^{(n)}}\|_{L^{p(\cdot)}(\mu)}.$$

Then observe that for each $\varepsilon > 0$ there is $j \in \mathbb{N}$ such that

$$\frac{1}{p(\cdot)} (\varphi(t))^{1-p(\cdot)} \geq \frac{1}{j} (\varphi(t) + \varepsilon)^{1-j}, \quad p(\cdot) \geq j, \quad \phi(1) + \varepsilon \leq 1.$$

This reads

$$(5.4) \quad \|1_{A_j^{(n)}}\|_{L^{p(\cdot)}(\mu)} \geq \|1_{A_j^{(n)}}\|_{L^j(\mu)} - \varepsilon$$

and further

$$(5.5) \quad \|1_{A_0^{(n)}}\|_{L^{p(\cdot)}(\mu)} \geq \limsup_{j \rightarrow \infty} \|1_{A_j^{(n)}}\|_{L^j(\mu)}.$$

Recall that

$$(5.6) \quad \|1_{A_j^{(n)}}\|_{L^j(\mu)} = (2^{-n})^{\frac{1}{j}} \mu(\{t \in [0, 1] : p(t) > j\})^{\frac{1}{j}} \rightarrow 1, \quad j \rightarrow \infty.$$

We made an additional assumption during the course of the proof that $1_{A_j^{(n)}}$ is included in the space. This assumption can be removed by observing that we may restrict the support of these functions to suitable sets $\{t : p(t) \leq p^{(n)}\}$, so that the positive-initial-value solutions become Lipschitz with a large constant and such that simultaneously (5.5) and (5.6) hold up to an extra ε . Thus $L_0^{p(\cdot)}$ contains c_0 almost isometrically.

The direction (4) \implies (1). Here we will follow the analogous argument in the setting of $\ell^{p(\cdot)}$ spaces. We will require the notions of upper p -estimate and lower q -estimate of Banach lattices. If X is a Banach lattice and $1 \leq p \leq q < \infty$ then the upper p -estimate and the lower q -estimate, respectively, are defined as follows:

$$\|\sum_{1 \leq i \leq n} x_i\| \leq \bigoplus_{1 \leq i \leq n}^p \|x_i\|,$$

$$\|\sum_{1 \leq i \leq n} x_i\| \geq \bigoplus_{1 \leq i \leq n}^q \|x_i\|,$$

respectively, for any vectors $x_1, \dots, x_n \in X$ with pairwise disjoint supports. These estimates involve multiplicative coefficients which are taken to be 1 in this treatment. We will apply the fact that a Banach lattice, which satisfies an upper p -estimate and a lower q -estimate for some $1 < p < q < \infty$ with constants 1 is both uniformly convex and uniformly smooth (with the respective power types), see [20, 1.f.1, 1.f.7].

Let $1 < p = \text{ess inf}_t p(t)$ and $\text{ess sup}_t p(t) = q < \infty$. We claim that $L^{p(\cdot)}$ satisfies the respective estimates for these p and q . To check the upper p -estimate, let f_k , $1 \leq k \leq n$, be disjointly supported functions in $L^{p(\cdot)}$. Observe that if X and Y satisfy the upper p -estimate, then $X \oplus_r Y$ satisfies it as well for $r \geq p$. Indeed,

$$\bigoplus_i^p \|(x_i, y_i)\|_{X \oplus_r Y} \geq \bigoplus_i^p \|x_i\|_X \oplus_r \bigoplus_i^p \|y_i\|_Y \geq \left\| \sum_i x_i \right\|_X \oplus_r \left\| \sum_i y_i \right\|_Y = \left\| \sum_i (x_i, y_i) \right\|_{X \oplus_r Y}$$

where we applied the direct sum norm twice, Proposition 1.5 and the upper p -estimate of X and Y . Thus, using this observation inductively on the semi-norms \mathcal{N} we obtain the statement by approximation.

Alternative route. By a simple argument using the definition of outer measure we see that each simple semi-norm can be approximated point-wise from below with other semi-norms of the type $\|\cdot\|_{(\dots(L^{p_1}(\mu_1) \oplus_{r_2} L^{p_2}(\mu_2)) \oplus_{r_3} \dots \oplus_{r_m} L^{p_m}(\mu_m))}$, $\text{ess inf}_t p(t) \leq r_i \leq \text{ess sup}_t p(t)$, such that only one of the functions f_k is supported on the support of a given μ_i . We may interpret the values of the semi-norms as norms of finite $\ell^{p(\cdot)}$ sequences

$$f \mapsto (|f|_{L^{p_1}(\mu_1)}, |f|_{L^{p_2}(\mu_2)}, \dots, |f|_{L^{p_m}(\mu_m)})$$

and then the supports of the sequences are disjoint for disjointly supported functions f_k . We apply the fact proved in [30] which states that for disjointly supported $\ell^{p(\cdot)}$ sequences we have the upper p -estimate for $p = \inf_t p_t$. From these considerations it follows that also disjointly supported $L^{p(\cdot)}$ functions satisfy the upper p -estimate for $p = \text{ess inf}_t p(t)$.

The argument for lower q -estimates is analogous. This concludes the proof. \square

Next, our aim is to build a kind of *universal* $L^{p(\cdot)}$ space. We will study a modification of Topologist's Sine Curve as follows:

$$p_0(t) = \frac{1}{1-t} \sin\left(\frac{1}{1-t}\right) + \frac{1}{1-t} + 1, \quad 0 \leq t < 1.$$

Theorem 5.3. *Let p_0 be as above. We consider a Borel measure μ on $[0, 1]$ given by $\frac{d\mu}{dm}(t) = |p'_0(t)|$ for m -a.e. t . Let $p: [0, 1] \rightarrow [1, \infty)$ be a C^1 -function, not constant on any proper interval and such that p' changes its sign finitely many*

times on each interval $[0, a] \subset [0, 1]$. Then there is an isometric linear embedding $L^{p(\cdot)} \rightarrow L^{p_0(\cdot)}(\mu)$ onto a projection band.

We note that p_0 satisfies the assumptions made about p and that the construction can be easily modified to accommodate the case where p is prescribed to be constant on some intervals.

Proof. By the assumptions we find a sequence of open subintervals $\Delta_n \subset [0, 1]$, $n \in \mathbb{N}$, with $\sup \Delta_n = \inf \Delta_{n+1}$ such that the sign of p' does not properly change on the intervals Δ_n . Moreover, we may assume that $|p'(x)| > 0$ for $x \in \bigcup_n \Delta_n$. We may choose this collection to be almost a cover in the sense that $m([0, 1] \setminus \bigcup_n \Delta_n) = 0$.

Now, p is monotone on each Δ_n . By the construction of p_0 we can find a sequence of open intervals $\Delta'_n \subset [0, 1]$, $n \in \mathbb{N}$, with $\sup \Delta'_n \leq \inf \Delta'_{n+1}$ such that there is a C^1 -diffeomorphism $T_n: \Delta_n \rightarrow \Delta'_n$ with $p|_{\Delta_n} = p_0 \circ T_n$.

By taking the union of the graphs of T_n , i.e. by 'gluing together' these mappings, we define a mapping T defined a.e. on $[0, 1]$, which has the property that $p(x) = p_0(T(x))$ for a.e. $x \in [0, 1]$.

Let us define a measure ν on $[0, 1]$ by $\frac{d\nu}{dm}(x) = |p'(x)|$. Both ν and μ can be thought as variation measures corresponding to p and p_0 , respectively. Thus it is easy to see that T is a ν - μ -measure-preserving mapping. Thus we get

$$\|1_{T([0,1])}f\|_{L^{p_0(\cdot)}(\mu)} = \|f \circ T\|_{L^{p(\cdot)}(\nu)} = \|g\|_{L^{p(\cdot)}(\nu)}$$

for $f \in L^{p_0(\cdot)}(\mu)$ such that $f \circ T = g \in L^{p(\cdot)}(\nu)$. Indeed, by using the absolute continuity of the solutions we observe that values of f outside $T([0, 1])$ do not influence the norm.

By making suitable identifications we may consider $L^{p(\cdot)}(\nu)$ as a subspace of $L^{p_0(\cdot)}(\mu)$, and disregarding the measure outside $T([0, 1])$ we may completely identify these spaces. This way we may apply Proposition 5.1 to observe that $G: L^{p(\cdot)}(\nu) \rightarrow L^{p_0(\cdot)}(\mu)$ given by

$$G(f)(t) = \left(\frac{d\nu}{dm} \right)^{-\frac{1}{p(t)}} (f \circ T^{-1})(t) = (|p'(t)|)^{\frac{1}{p(t)}} (f \circ T^{-1})(t) \quad \text{if } t \in T([0, 1]),$$

and $G(f)(t) = 0$ otherwise, defines the required isometry. \square

The following result provides some information on an invariant involving the exponents.

Theorem 5.4. *Consider spaces $L^{p(\cdot)}(\mu)$ and $L^{p^*(\cdot)}(\mu)$. Let us consider functions $f \in L^{p(\cdot)}(\mu)$, $Jf \in L^{p^*(\cdot)}(\mu)$ such that $f \neq 0$ a.e. and $\varphi_f^{p(\cdot)}(\cdot) = \varphi_{Jf}^{p^*(\cdot)}(\cdot)$. Then*

$$\iota_f := \frac{\varphi'_{Jf}}{\varphi'_f} \frac{\varphi_f}{\varphi_{Jf}} = \frac{d \log \varphi_{Jf}}{d \log \varphi_f}$$

satisfies

$$(5.7) \quad \iota_f = \frac{p(\cdot)}{p^*(\cdot)}$$

a.e., and, consequently, does not depend on the particular choice of f . Moreover, if f is admissible as above, then

$$\frac{d\mu}{dm} = (\iota + 1) \varphi'_f |f|^{-\iota-1} \varphi_f^\iota.$$

Proof. The first part of the statement follows immediately from the equality (4.8) for $L^{p(\cdot)}(\mu)$ type spaces. Indeed, this, in turn, holds similarly as in the usual case since the densities cancel each other.

In the latter part we use the preceding observation together with the fact $p(\cdot) = \iota(\cdot) + 1$ and (5.1). \square

5.3. Extension to several dimensions. The definition of the $L^{p(\cdot)}$ spaces appears to be fundamentally 1-dimensional, being essentially on ODE, and a priori it is not clear if it extends naturally to the n -dimensional setting, i.e. spaces of the type $L^{p(\cdot)}(\Omega)$ with $\Omega \subset \mathbb{R}^n$. This appears as a weakness in the definition of the spaces under investigation.

We wish to remove this obstruction and indicate a possible extension to several dimensions. By a similar heuristical motivation as we have seen in the 'derivation' of the differential equation with φ' , one finds the following way of defining norms in domains $\Omega \subset \mathbb{R}^n$, instead of the unit interval. Suppose that $p: \Omega \rightarrow [0, \infty)$ and $f: \Omega \rightarrow \mathbb{R}$ are measurable. Our aim is to define

$$\varphi(x) = \|1_{y \leq x}(y)f(y)\|_{L^{p(\cdot)}(\Omega)}, \quad x = (x_1, \dots, x_n) \in \Omega$$

where $y \leq x$ is the partial order of coordinate-wise dominance. Here φ is going to be absolutely continuous on every line (ACL) parallel to coordinate axes.

Let us study the equality

$$(5.8) \quad \partial_i \varphi(x) = \int_{\substack{y \leq x \\ y_i = x_i}} \frac{|f(y)|^{p(y)}}{p(y)} \varphi^{1-p(y)} dm_{n-1}, \quad \text{for a.e. } x \in \mathbb{R}^n.$$

The convenient feature about (5.8) is that if $\Omega \subset \{y \in \mathbb{R}^n: z_0 \leq y\}$ for some $z_0 \in \mathbb{R}^n$ then (5.8) defines a function φ similarly as in the 1-dimensional case for an analogous limit process on the initial condition $\varphi(z_0) = 0^+$ and suitable f . (The key idea in analyzing the accumulation of the values of φ is that in adding small new blocks

$$\{y: y_j = x_j, j \neq i, y_i \in [x_i, x_i + \Delta]\}$$

they 'interact' with the big block $\{y \in \mathbb{R}^n: y \leq x\}$ in an essential way but the mutual interaction of the small blocks vanishes rapidly as Δ tends to 0.)

Let us study the properties of (5.8). The integral can be seen as a $(n-1)$ -fold integral and by applying the fundamental theorem of calculus $(n-1)$ -times (at each step disregarding an m_1 -null set of coordinates), we obtain that

$$(5.9) \quad \partial_1 \dots \partial_{i-1} \partial_{i+1} \dots \partial_n \partial_i \varphi(x) = \frac{|f(x)|^{p(x)}}{p(x)} \varphi^{1-p(x)}, \quad \text{for a.e. } x \in \mathbb{R}^n.$$

The $(n-1)$ -fold integral can be written in any order by Fubini's theorem, and, consequently, so can the derivatives ∂_j , $j \neq i$. Thus, by looking at the right hand side we observe immediately that the order of taking the derivatives on the left hand side does not matter a.e.

Therefore we have

$$(5.10) \quad \partial_1 \dots \partial_n \varphi(x) = \frac{|f(x)|^{p(x)}}{p(x)} \varphi^{1-p(x)}, \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Going backwards by integration we observe that the above PDE in fact characterizes an ACL φ described by (5.8).

This suggests defining a norm corresponding to any given open set $\Omega \subset \mathbb{R}^n$ and any Borel measure $\mu \ll m$ on it as follows

$$(5.11) \quad \|f\|_{L^\Upsilon(\Omega, \mu)} = \inf_{\psi} \sup_{x \in \Omega} \psi(x)$$

where the infimum is taken over all ACL functions $\psi: \Omega \rightarrow (0, \infty)$ satisfying

$$\partial_1 \dots \partial_n \psi(x) = \frac{d\mu}{dm}(x) \Upsilon(\psi(x), |f(x)|, x), \quad \text{for a.e. } x \in \Omega$$

and Υ satisfies similar structural conditions as above.

We leave studying the properties of these function spaces to future research. Another such research thread is to look at functionals of the type $F_p(f) = \|g\|_{L^{r(\cdot)}(0, \infty)}$ where p is a varying exponent on Ω and $r: (0, \infty) \rightarrow [1, \infty)$ is an absolutely continuous non-decreasing function and $h: \Omega \rightarrow [1, \infty)$ is a m_n -measure-preserving function such that $h^{-1}r^{-1}((r_1, r_2)) = \{x \in \Omega: r_1 < p(x) < r_2\}$ for all $1 \leq r_1 < r_2 < \infty$. Here

$$g(t) = \left(\int_{h^{-1}(t)} |f(s)|^{r(t)} dH_{n-1}(s) \right)^{\frac{1}{r(t)}}$$

for a.e. $t \in (0, \infty)$ where H_{n-1} is the $(n-1)$ -dimensional Hausdorff measure (cf. [21]) and suitable regularity conditions on $p(\cdot)$ must be imposed. This kind of functional is invariant under suitable measure-preserving rearrangements of Ω . The author acknowledges Jan Lang for inspiring discussions involving this approach. Some further extensions are discussed shortly.

6. DISCUSSION

Above we observed that the $L^{p(\cdot)}$ class is a Banach space if $p(\cdot)$ is essentially bounded. The same conclusion holds if $1_{[0,t]}p(\cdot)$ is essentially bounded for all $0 < t < 1$. Even if the class is not linearly closed, the subclass $L_0^{p(\cdot)}$, roughly consisting of those restricted functions f making $p(\cdot)$ appear almost bounded, is always a Banach space.

One possible extension to improve the stability of the linear structure is to use positive initial values x_0 such that the solution φ_{f, x_0} exists, and then take the infimum of all the values $\varphi_{f, x_0}(1)$, similarly as in (5.11).

It turned out in Proposition 3.3 that our ODE-driven norms are equivalent to Nakano $L^{p(\cdot)}$ norms for the same $p(\cdot)$. However, looking at the difference of the behavior of these norms under rearrangements of the measure space we concluded that these types of norms do not coincide isometrically.

We note that it is straightforward to define a version of varying exponent Sobolev spaces from the norms investigated here.

The basic ODE studied here is in an obvious way positively homogeneous (in the sense of norms) because of the cancellation due to the terms $|f(t)|^{p(t)}$ and $(\varphi(t))^{-p(t)}$, and it is ‘almost separable’ as a differential equation, being separable in the constant p case. Note that the ODE is linear exactly in the case $p \equiv 1$. Although our ODE is typically not linear, the above-mentioned cancellation appears to contribute towards linearity; a perfect cancellation (for suitable f) should produce a form $\varphi' = a \frac{\varphi(t)}{p(t)}$. Thus the basic ODE appears a rather well-behaved one and we do not know if it can be conveniently placed in the taxonomy of differential equations or if the corresponding theory can be applied to analyze these spaces.

We have usually written φ'_f as the multiple of the ‘Nakano part’, $\frac{|f(t)|^{p(t)}}{p(t)}$, and the compensator term $(\varphi(t))^{1-p(t)}$ which influences the process towards the steady state $\varphi_f = 1$. Thus the Nakano norm is computed much like our ODE driven norm, except that the steady state value is used in place of φ_f .

As mentioned in the introduction, the approach taken to function space norms here is inductive or local, rather than global. One can also conceive a ‘left-handed’ version of the L^Υ spaces where the process travels backwards, i.e.

$$\varphi'_f(t) = \Upsilon(\varphi(t), |f(1-t)|, t), \quad \text{for a.e. } t \in [0, 1].$$

Hopefully the lack of rearrangement-invariance of the spaces here does not appear a very unnatural feature. There is a standard trick to symmetrizing a norm with respect to a given group of transformations, for example, one may define a new norm by

$$|||f||| = \sup_T \|f \circ T\|_{p(\cdot)}$$

for suitable functions f where the supremum is taken over measure-preserving transformations $T: [0, 1] \rightarrow [0, 1]$ (possibly with $p = p \circ T$ a.e.). However, some mathematicians may find the example $L^r(0, \frac{1}{2}) \oplus_p L^p(\frac{1}{2}, 1)$, perceived as a function space, to be convincing enough, so that rearrangement-invariance is not an absolutely necessary property of function spaces, although the L^p spaces with a constant p , as well as many other spaces enjoy this property. In the left-handed case the corresponding example is $L^r(0, \frac{1}{2}) \oplus_r L^p(\frac{1}{2}, 1)$.

In applied sciences the Hilbertian norm has of course numerous uses but also other L^p norms are widely used. For example, larger values of p can be used in penalizing especially large deviations in (curve) fitting. This has an interesting side effect; although the norm distinguishes between the maximal absolute values, it does not distinguish so much between the *measures* of the *supports* of the (near)maximum value. That is, if there are already some recorded deviations of certain magnitude, then other such observations do not change the norm that much. This phenomenon can be analyzed more formally by using the framework of this paper, the basic ODE tells exactly how the accrued norm accumulation (e.g. after initial observations) slows down the future norm accumulation. Perhaps the ODE (or just the p -norms) could be also used in modeling the economics of innovation where the first occurrences of so far highest achievements have a disproportionate impact on the wealth (of an individual or society). For p close to infinity this metaphor becomes the most pronounced, with a ‘winner takes it all’ logic, the norm accumulation depends at each moment of time on the highest value until then (actually only on the first attainment of that value). Turning away from potential applications, we will discuss the infinite p case next.

In this paper we have only studied absolutely continuous solutions φ to ODEs. It appears that one could extend the class of the norms studied here by admitting φ' to be a proper distribution. This could allow the exponent p to attain the value ∞ . We leave this for future work. On the other hand, the ‘approximate monotonicity’ of the semi-norms with respect to the exponent suggests the following natural extension:

$$\|f\|_{L^{p(\cdot)}} := \limsup_{\tilde{p} \in L^\infty, \tilde{p} \nearrow p} \|f\|_{L^{\tilde{p}}}, \quad p: [0, 1] \rightarrow [1, \infty]$$

(lim sup over a net). Let us still take a brief look at the asymptotics of the equation

$$\frac{d\varphi}{dt} = \frac{|f(t)|^p}{p} \varphi(t)^{1-p}$$

as p tends to ∞ . It is clear that if $|f(t)| \leq \varphi(t)$, then the limit is 0. This, of course, is the anticipated outcome when thinking how the hypothetical norm accumulation function φ should behave in the L^∞ case. If $|f(t_0^+)| = \varphi(t_0) + a$, $a > 0$, then we would expect in the L^∞ case that

$$(6.1) \quad d\varphi = a\delta_{t_0}$$

holds at t_0 . The following calculation

$$d\varphi = (\varphi^p(t_0) + \Delta(\varphi(t_0) + a)^p)^{\frac{1}{p}} - \varphi(t_0) \xrightarrow{p \rightarrow \infty} a, \quad \forall \Delta > 0$$

indeed appears to point to the direction of (6.1).

So far we have applied the symbol \boxplus_p as a short hand notation only. Next we will briefly look into the relationship between the algebraic structure of this operation and the basic ODE studied above.

We observe that $h_{z_1, p}: \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ given by

$$h_{z_1, p}(z, \Delta) = (z^p + \Delta z_1^p)^{\frac{1}{p}}, \quad p \in \mathbb{C} \setminus \{0\}$$

defines a group action where we consider \mathbb{C} in the Δ -coordinate with its additive structure and the other copies of the complex plane as sets.

This way we may define the ‘left roots’ corresponding to the $\cdot \boxplus_p t$ operation as follows:

$$r \boxplus_p \left[\bigoplus_{1 \leq i \leq n}^p \right] t = s \boxplus_p t$$

where

$$r = (h_{t, p}(\cdot, -n^{-1}) \circ \dots \circ h_{t, p}(\cdot, -n^{-1}))(s) \quad (n - \text{times}).$$

The formula of φ' essentially provides us with the formal derivative of the semi-group $(S_t)_{t \geq 0}$ given by $S_t(x) = (x^p + ty^p)^{\frac{1}{p}}$, y being a constant. Namely,

$$\frac{dS_t}{dt} = \frac{d}{dt} (x^p + ty^p)^{\frac{1}{p}}, \quad \left. \frac{dS_t}{dt} \right|_{t=0} = \frac{y^p}{p} x^{1-p}.$$

However, the infinitesimal generator does not exist for $x = 0$, since

$$\frac{d}{dt} t^{\frac{1}{p}} = \frac{t^{\frac{1}{p}-1}}{p} \rightarrow \infty, \quad t \rightarrow 0^+$$

unless $p = 1$, the case where the accumulation of $\varphi(t)$ depends on $|f(t)|$ solely. This appears to be related to the fact that the ODEs studied here are not very stable around the initial value 0.

There are obviously many open questions related to these $L^{p(\cdot)}$ spaces. It is known that the Luxemburg norms do not coincide isometrically with the norms introduced here, but we do not know if something can be said about the equivalence of the norms. In fact, we do not know whether $L^{p_1(\cdot)}$ and $L^{p_2(\cdot)}$ are isomorphic whenever there is a measure-preserving mapping $T: [0, 1] \rightarrow [0, 1]$ such that $p_1 = p_2 \circ T$.

We would like to see a classification for the surjective linear isometries acting on $L^{p(\cdot)}$ spaces, or at least find some examples of $p(\cdot)$ such that these isometries have the trivial form, i.e. $f \mapsto \sigma f$ with $\sigma: [0, 1] \rightarrow \{-1, 1\}$ measurable (cf. [11],

[14]). We suspect that this is the case when $p(\cdot)$ is not a constant (a.e.) on any subinterval $[a, b] \subset [0, 1]$.

We do not know if $L_0^{p(\cdot)}$ is strictly convex (cf. [10], [13], [16], [17], [18]). We suspect that for $L^{p(\cdot)}$ spaces superreflexivity is equivalent to being Asplund.

We do not know any criteria for the uniform convexity or uniform smoothness of the norms (or reflexivity, RNP, Asplund, LUR) for a general Υ . Related to this problem, we would like to call attention to the duality pairing perceived as an ODE in this context. The author considers this to be one of the most interesting future avenues on these spaces. For instance, what is the relationship in the reflexive case between Υ -solutions φ_f and $\varphi_{J(f)}^*$ (if applicable, here J is the duality mapping, cf. (4.8))? We do not know if the superreflexivity of L^Υ is characterized by the condition that there is a compact interval $[a, b] \subset (0, \infty)$ such that the range of ι_f lies on $[a, b]$ for any $f \in L^\Upsilon$, $f \neq 0$ a.e. This question has two natural versions for $L^{p(\cdot)}$ spaces, the above general one, and a one where we restrict to looking at the functions with $\|Jf\|_{p(\cdot)}^* = \|Jf\|_{p^*(\cdot)}$ (one must then somehow ensure that such regular functions f are abundant enough to sufficiently capture the behavior of the norm). We suspect that ι_f provides a useful ‘local type’ of the space, even if the space fails to be superreflexive. The above index may provide also some quantitative information, such as the Boyd indices, p -convexity, (co)type or power type of the modulus of convexity/smoothness. The author expects the duality theory of Banach spaces in some part to reflect to a kind of reciprocal involving differential equations.

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